Notes on Algebraic Geometry

Yang Zhang

Bonn, Germany

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Mathematisch-Naturwissenschaftliche Fakultät der

Rheinischen Friedrich-Wilhelms-Universität Bonn

Contents

Preface

The notes are aimed to be a summary of all contents in foundations of algebraic geometry, which are based on the lecture courses Algebraic Geometry I and Algebraic Geometry II at the Universität Bonn. The whole story is separated into five parts: Schemes, Modules, Smoothness, Cohomology and Curves.
To make stuffs well categorized I switched the order of some contents, so the order of contents in notes actually differ from the usual organized way in Bonn. For example the theory of projective schemes will be separated into the construction of projective space, the (very) ample line bundles, Serre duality, projective curves and Riemann-Roch on it, and so on. So **the order of all contents may confuse the beginners**.
The notes are also not self-contained. Indeed, **I omitted all the proofs of easy propositions**. The definition of "easy" here is, if one has gone over the whole courses AG I, AG II, he/she should be able to figure out a proof for this proposition within ten minutes. But for beginners it will take sometimes hours. **I also did not include difficult proofs of some big theorems**, there might be only a sketch of proof which just gives the intuition and frame. But A reference will always be given where one can find the whole proof. The only proofs I typed down are those I didn't understand when I was learning the stuffs (I type them down and try to understand them thoroughly, and they may also be hard points for other intermeidates), and those proofs the techniques in which are of great importance and can be applied frequently. Moreover, I only give references of commutative algebra facts instead of a proof when they are crucial for a proposition, since I believe that these are just technical details and do no good to the build-up of the intuition in algebraic geometry. Based on these three points, I believe **these notes are not suitable for AG beginners, but**

might be very helpful as toolbook after one finishes the whole journey in the foundations of algebraic geometry.

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Chapter 1

Schemes

1.1 General Sheaf Theory

The Motivation of sheaves is that we want to understand how local functions on a manifold "glue" together to be a global function. For instance, consider the complex manifold $\mathbb{CP}^1$ with homogeneous coordinate $[z, w]$. We have the cover $U_0 = \{w \neq 0\}$ and $U_1 = \{z \neq 0\}$, which are homeomorphic to $\mathbb{C}$. Then a meromorphic function on U_0 is defined by $f([z, w]) = F(\frac{z}{w})$ for some meromorphic function F on $\mathbb{C}$. Similarly we have $g([z, w]) = G(\frac{w}{z})$ on U_1. Note we can restrict f and g onto $U_0 \cap U_1$ to get two meromorphic functions on it. If $F(x) = G(x^{-1})$, then f and g agree on the overlap and define a global meromorphic function.
In algebraic geometry, things work almost in the same way, just by replacing meromorphic functions with polynomials. So it is worth spending a whole section to have a lesson in sheaf theory.

Definition 1.1.1. A **presheaf** $\mathcal{F}$ on a topological space X is a functor

$$\mathcal{F} : \mathrm{Ouv}_X^{\mathrm{op}} \longrightarrow \mathcal{C},$$

where Ouv_X is the category whose objects are open sets in X and arrows are inclusions of open sets. We call $\mathcal{F}(U \hookrightarrow V) =: \mathrm{res}_{\mathrm{U}}^{\mathrm{V}}$ the restriction from $\mathcal{F}(V)$ to $\mathcal{F}(U)$. We also write $f|_U$ for $\mathrm{res}_{\mathrm{U}}^{\mathrm{V}}(f)$.
A **sheaf** $\mathcal{F}$ is a presheaf satisfying the addtional conditions:

1. (locality) For $s, t \in \mathcal{F}(U)$, if there exists an open cover $U = \bigcup_i U_i$ such that $s|_{U_i} = t|_{U_i}$ for all i, then $s = t$.

2. (gluing) If $U = \bigcup_i U_i$ is an open cover of an open and $s_i \in \mathcal{F}(U_i)$ are functions on U_i satisfying $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there exists an $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all i.

Remark 1.1.2. If $\mathcal{F}$ is a sheaf, then the glued element s in 2. is unique by 1..

Convince yourself that this definition really makes sense with the example $\mathbb{CP}^1$, where one maps an open to the set of all meromorphic functions on it.

Definition 1.1.3. We also write $\Gamma(U, \mathcal{F})$ for $\mathcal{F}(U)$, which turns out to be more useful in the cohomology theory.

If we have a space with a nice basis, then the information of the sheaf on the basis is already enough to recover the whole sheaf, asserted as follows.

Proposition 1.1.4. *If X admits a basis $\mathcal{B}$ which is stable under intersections, then given a functor $\mathcal{F} : \mathcal{B}^{\mathrm{op}} \to \mathcal{C}$ (the arrows in $\mathcal{B}$ are inclusions, and note that $\mathcal{B}$ is a full subcategory of* Ouv_X*) there exists a unique presheaf $\mathcal{G}$ on X valued in $\mathcal{C}$ such that $\mathcal{G}|_{\mathcal{B}} = \mathcal{F}$.*

Remark 1.1.5. if the "presheaf" on $\mathcal{B}$ satisfies locality and glueing, then the costructed presheaf $\mathcal{G}$ is already a sheaf.

Definition 1.1.6. Let $\mathcal{F}, \mathcal{G}$ be presheaves valued in $\mathcal{C}$. A **morphism of presheaves** $\phi : \mathcal{F} \to \mathcal{G}$ is just a natural transformation of functors.

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r, "\phi_V"] \arrow[d, "\mathrm{res}^V_U"'] & \mathcal{G}(V) \arrow[d, "\mathrm{res}^V_U"] \\
\mathcal{F}(U) \arrow[r, "\phi_U"'] & \mathcal{G}(U)
\end{tikzcd}$$

A **morphism of sheaves** is just a morphism of the underlying presheaves.

Definition 1.1.7. Let $\mathcal{F}$ be a presheaf on X. The **stalk** of $\mathcal{F}$ at $x \in X$ is defined as

$$\mathcal{F}_x := \operatorname*{colim}_{x \in U, U \subseteq X} \mathcal{F}(U)$$

Note that the index of colimit is filtered, so we have a nice representation of elements in $\mathcal{F}_x$.

Proposition 1.1.8. *Let $\mathcal{F}, \mathcal{G}$ be sheaves on X, $\phi : \mathcal{F} \to \mathcal{G}$ a morphism of sheaves. Then $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective (resp. bijective), if the induced morphism on stalks $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective (resp. bijective) for all $x \in U$.*

Lemma 1.1.9. *Succeeding the notations above, let $\psi : \mathcal{F} \to \mathcal{G}$ be another morphism of sheaves. Then $\phi = \psi$ if and only if $\phi_x = \psi_x$ for all $x \in X$.*

Remark 1.1.10. This proposition shows that isomorphisms of sheaves can be checked on stalks. However surjectivity on stalks does not always imply surjectivity on each open. We define a morphism of sheaves to be injective (resp. surjective, bijective) if it is injective (resp. surjective, bijective) on stalks. They are precisely monomorphisms (resp. epimorphisms, isomorphisms) if the value category is abelian.

The next proposition tells us that we do not need to worry to much about presheaves.

Proposition 1.1.11. *Let $\mathcal{F}$ be a presheaf. Then there exists a sheaf $\tilde{\mathcal{F}}$, unique up to unique isomorphism, called the **sheafification** of $\mathcal{F}$, together with a map of presheaves $\iota : \mathcal{F} \to \tilde{\mathcal{F}}$, satisfying the universal property: For any morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$ where $\mathcal{G}$ is a sheaf, there exists a unique morphism of sheaves $\tilde{\phi} : \tilde{\mathcal{F}} \to \mathcal{G}$ making the following diagram commute:*

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\iota"] \arrow[d, "\phi"'] & \tilde{\mathcal{F}} \arrow[dl, dashed, "\exists! \tilde{\phi}"] \\ \mathcal{G} & \end{tikzcd}$$

Sheaves are often used to describe objects that "live" on a topological space, so we want also to "transport" sheaves onto another space if we have a map between spaces. Thus we now define the pullback and pushforward of sheaves.

Definition 1.1.12. Let X, Y be topological spaces, $f : X \to Y$ a continuous map.

1. For a sheaf $\mathcal{F}$ on X we define the **pushforward** of $\mathcal{F}$ along f to be the sheaf on Y:

$$f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$$

2. For a sheaf $\mathcal{G}$ on Y we define the **pullback** of $\mathcal{G}$ along f to be the sheafification of the presheaf on X:

$$f^{-1}\mathcal{G}(U) = \operatorname*{colim}_{f(U)\subseteq V} \mathcal{G}(V)$$

Note the morphisms of sheaves can also be pushed forward or pulled back in a natural sense. Thus one should consider the pushforward and pullback as functors between the category of $\mathcal{C}$-valued sheaves on X and on Y.

Proposition 1.1.13. *Succeed the notations above.*

1. *For $x \in X$ we have*

$$f^{-1}\mathcal{G}_x = \mathcal{G}_{f(x)}.$$

2. *(Pullback-Pushforward Adjunction) There is a natural bijection*

$$\mathrm{Hom}_{Sh_X}(f^{-1}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{Sh_Y}(\mathcal{G}, f_*\mathcal{F}),$$

i.e. f^{-1} and f_* are adjoints.

Finally we discuss an important case: spaces with "structure" sheaf. The idea is, to understand an geometric object we need to understand the functions living on it. In many situations, the sheaf of local functions has a ring structure (e.g. the sheaf of meromorphic functions on $\mathbb{CP}^1$), i.e. they are sheaf of rings.

Definition 1.1.14. A **ringed space** $(X, \mathcal{O}_X)$ is a topological space X, endowed with a sheaf of rings $\mathcal{O}_X : \mathrm{Ouv}_X^{\mathrm{op}} \to \mathbf{Rings}$, called the structure sheaf. A **locally ringed space** is a ringed space whose stalks are all local rings.

Definition 1.1.15. A **morphism of ringed spaces** $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f : X \to Y$ on the underlying topological space, together with a morphism of sheaves $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. A **morphism of locally ringed spaces** is a morphism of ringed spaces such that the induced map on stalks $f_x^\sharp : (f^{-1}\mathcal{O}_Y)_x \cong \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local ring homomorphism.

Remark 1.1.16. By pullback-pushforward adjunction, to give $f^\sharp$ is the same as to give a morphism of sheaves $f_\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X$.

Convince yourself that this definition makes sense with the example of two smooth manifolds $\phi : \mathcal{M}_1 \to \mathcal{M}_2$. $\phi_\flat$ here is given by the pullback:$f \mapsto f \circ \phi$.

1.2 Affine Schemes

We start with a generalization of the vector space k^n, where k is a field. Recall the Nullstellensatz:

Theorem 1.2.1 (Hilbert's Nullstellensatz)**.** *Let k be an algebraically closed field, then all maximal ideals in $k[X_1, \ldots, X_n]$ are in the form $(X_1 - a_1, \ldots, X_n - a_n)$.*

Proof. [5] Theorem 1.7. $\square$

One sees from the Nullstellensatz that there is a bijection between the points $(a_1, \ldots, a_n)$ and the maximal ideals $(X_1 - a_1, \ldots, X_n - a_n)$ in the polynomial ring, which inspires us to study the spectrum of a ring defined as follows.

Definition 1.2.2. We define the **spectrum** of a ring A as the set of all prime ideals:

$$\operatorname{Spec} A := \{\mathfrak{p} \subset A | \mathfrak{p} \text{ prime}\}$$

We define the **vanishing locus** of an ideal $I \subseteq A$ as:

$$V(I) := \{\mathfrak{p} \in \operatorname{Spec} A | I \subseteq \mathfrak{p}\}$$

Remark 1.2.3. Without further mention, all rings in the notes are commutative with 1.

Proposition 1.2.4. *We have*

$$\bigcap_i V(I_i) = V\left(\sum_i I_i\right).$$
$$V(I_1) \cup V(I_2) = V\left(I_1 \cap I_2\right).$$

Proof. [5] Proposition 3.6. $\square$

Definition 1.2.5. The vanishing loci of ideals as closed sets define a topology on $\operatorname{Spec} A$, which we call **Zariski Topology**.

Later we will endow $\operatorname{Spec} A$ with a sheaf containing its algebraic structure, but now let's develop some topological properties first.

Proposition 1.2.6. Spec *can be extended to a functor from* $\mathbf{Rings}^{\mathrm{op}}$ *to* **Top**, *sending a ring* A *to* $\operatorname{Spec} A$, *and a homomorphism* $f : A \to B$ *to a continuous map* $\operatorname{Spec} f : \operatorname{Spec} B \to \operatorname{Spec} A, \operatorname{Spec} f(\mathfrak{p}) := f^{-1}(\mathfrak{p})$

Lemma 1.2.7. $\operatorname{Spec} A$ *is* T_0.

Definition 1.2.8. Let $f \in A$ be a ring element. We call $D(f) := \operatorname{Spec} A \backslash V(f)$ a **principal open** (or a distinguished open).

Lemma 1.2.9. *The set of all principal opens in* $\operatorname{Spec} A$ *forms a basis of Zarisiki Topology and is stable under intersections. In particular,* $D(f) \cap D(g) = D(fg)$

Proposition 1.2.10. $\operatorname{Spec} A$ *is quasi-compact.*

Proof. By the Alexander subbasis lemma, it suffices to prove that for each open covering of the form $\operatorname{Spec} A = \bigcup_i D(f_i)$, there exists a finite subcover. Now $\operatorname{Spec} A = \bigcup_{i \in I} D(f_i)$ is equivalent to $\bigcap_{i \in I} V(f_i) = \emptyset$, which is equivalent to that the ideal generated by f_i contains 1. That means that there exists $f_1, \ldots, f_n$ among all f_i and $a_1, \ldots, a_n$ in A such that $\sum_{j=1}^n a_j f_j = 1$. By the same argument we see $\operatorname{Spec} A = \bigcup_{j=1}^n D(f_j)$ ☐

Proposition 1.2.11. *The principal open* $D(f)$ *in* $\operatorname{Spec} A$ *is homeomorphic to* $\operatorname{Spec} A[f^{-1}]$. *The closed locus* $V(I)$ *is homeomorphic to* $\operatorname{Spec} A/I$.

Corollary 1.2.12. *Let* $A_{red} := A/Nil(A)$ *be the reduced ring of* A. *Then* $\operatorname{Spec} A$=$\operatorname{Spec} A_{red}$.

Lemma 1.2.13. *Let* ϕ *be a ring homomorphism* $A \to B$. *Then the image of* $D(f) \subseteq \operatorname{Spec} B$ *under* $\operatorname{Spec} \phi$ *is* $D(\phi(f)) \subseteq \operatorname{Spec} A$.

Lemma 1.2.14. $\operatorname{Spec} A$ *is* ***irreducible*** *(i.e. it can't be written as the union of two proper closed subsets) if and only if* $Nil(A)$ *is prime. This is in particular the case when* A *is an integral domain.*

Proposition 1.2.15. *Every irreducible closed set in* $\operatorname{Spec} A$ *has a unique generic point.*

Theorem 1.2.16 (Generalized Hiblert's Nullstellensatz)**.** *Let A be a ring. Then we have two bijections:*

1. *There is a bijection between the closed subsets of* $\operatorname{Spec} A$ *and the radical ideals in A, by sending $V \subseteq \operatorname{Spec} A$ to $I(V) := \bigcap_{\mathfrak{p} \in V} \mathfrak{p}$, and $I \subseteq A$ to $V(I)$.*

2. *There is a bijection between the irreducible closed subsets of* $\operatorname{Spec} A$ *and the prime ideals in A, by sending $V \subseteq \operatorname{Spec} A$ to its generic point, and $\mathfrak{p} \subseteq A$ to $V(\mathfrak{p})$.*

Proof. [5] Theorem 1.17. ☐

Lemma 1.2.17. *If A is a Noetherian ring, then* $\operatorname{Spec} A$ *is a* ***Noetherian topological space*** *(i.e. every descending chain of closed subsets stabilizes).*

Remark 1.2.18. The converse does not hold in general. Consider $A := k[X_1, X_2, \dots]/(X_1^2, X_2^2, \dots)$. We see that $\operatorname{Spec} A$ has only one point $(X_1, X_2, \dots)$, but this ideal is not finitely generated.

Lemma 1.2.19. *The dimension of* $\operatorname{Spec} A$ *is equal to the Krull dimension of A. (We define the* ***dimension*** *of a T_0-space X to be the maximal length of specializations $x_1 \rightsquigarrow x_2 \rightsquigarrow \cdots \rightsquigarrow x_n$)*

We now start to talk about the algebraic structure of $\operatorname{Spec} A$

Definition 1.2.20. We have a contravariant functor from the basis category of all principal opens to **Rings**:

$$\begin{aligned} \mathcal{F} : \mathcal{B}^{\mathrm{op}} &\to \mathbf{Rings}, \\ D(f) &\mapsto A[f^{-1}] \end{aligned}$$

This functor satisfies locality and gulability, hence defines a sheaf of rings $\mathcal{O}_{\operatorname{Spec} A}$ on $\operatorname{Spec} A$. Thus $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ becomes a ringed space. We call a ringed space an **affine scheme** if it is isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some ring A. We denote since now the ringed space directly with $\operatorname{Spec} A$, and its underlying topological space with $|\operatorname{Spec} A|$.

Remark 1.2.21. The structure sheaf $\mathcal{O}_{\operatorname{Spec} A}$ has the explicit form

$$\Gamma(U, \mathcal{O}_{\operatorname{Spec} A}) = \lim_{D(f) \subseteq U} A[f^{-1}].$$

In particular, $\Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) = A$, since $\operatorname{Spec} A = D(1)$.

Lemma 1.2.22. $\operatorname{Spec} A$ *is a locally ringed space, and* $\mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}} \cong A_{\mathfrak{p}}$.

Definition 1.2.23. We define the category of affine schemes **AffSch** to be the full subcategory of the category of locally ringed spaces, whose objects are all affine schemes.

Theorem 1.2.24. Spec *can be extended to a functor* $\mathbf{Rings}^{\mathrm{op}} \to \mathbf{AffSch}$*, by sending a ring* A *to* $\operatorname{Spec} A$*, and a homomorphism* $f : A \to B$ *to* $\operatorname{Spec} f := (|\operatorname{Spec} f|, \operatorname{Spec} f_\flat)$*, defined as*

$$\begin{aligned} |\operatorname{Spec} f| : |\operatorname{Spec} B| &\to |\operatorname{Spec} A|, \\ \mathfrak{p} &\mapsto f^{-1}(\mathfrak{p}). \\ \operatorname{Spec} f_\flat : \mathcal{O}_{\operatorname{Spec} A} &\to f_* \mathcal{O}_{\operatorname{Spec} B} : \end{aligned}$$

$$\begin{tikzcd}
\mathcal{O}_{\operatorname{Spec} A}(D(s)) \arrow[r] \arrow[d, "\cong"'] & \mathcal{O}_{\operatorname{Spec} B}(D(f(s))) \arrow[d, "\cong"] \\
A[s^{-1}] \arrow[r, "{f[s^{-1}]}"'] & B[f(s)^{-1}]
\end{tikzcd}$$

Remark 1.2.25. We have actually only defined $\operatorname{Spec} f_\flat$ only on principal opens. But this can be extended to arbitrary opens using the universal property of limits:

$$\begin{tikzcd}
& \lim_{D(f) \subseteq U} \mathcal{O}_A(D(f)) \cong \mathcal{O}_A(U) \arrow[dl] \arrow[dr] \arrow[dd, dashed, "\exists!"] & \\
\mathcal{O}_A(D(f_i)) \arrow[dd] & & \mathcal{O}_A(D(f_j)) \arrow[dd] \\
& \lim_{D(f) \subseteq U} f_* \mathcal{O}_B(D(f)) \cong \mathcal{O}_B(U) \arrow[dl] \arrow[dr] & \\
f_* \mathcal{O}_B(D(f_i)) & & f_* \mathcal{O}_B(D(f_j))
\end{tikzcd}$$

The functor above is actually an equivalence of categories. Instead of proving this we will show a more general result.

Theorem 1.2.26. *Let $(X, \mathcal{O}_X)$ be a locally ringed space,* $\operatorname{Spec} B$ *an affine scheme. Then there is a natural bijection*

$$\operatorname{Hom}((X, \mathcal{O}_X), \operatorname{Spec} B) \to \operatorname{Hom}(B, \Gamma(X, \mathcal{O}_X)).$$

Proof. [3] Tag 01I1 □

Corollary 1.2.27. *The functor* Spec *is an equivalence of categories between* **Rings** *and* **AffSch**.

Definition 1.2.28. We define the affine space $\mathbb{A}_k^n$ over a field k to be $\mathbb{A}_k^n := \operatorname{Spec} k[X_1, \ldots, X_n]$.

1.3 Schemes and Morphisms

We now come to the definition of the central objects in algebraic geometry.

Definition 1.3.1. A **scheme** X is a locally ringed space with an open cover $X = \bigcup_i U_i$, such that each U_i is an affine scheme. A **morphism of schemes** is just a morphism of locally ringed spaces.

Proposition 1.3.2. *Let X be a scheme, and $U \subseteq X$ be a locally ringed subspace. Then U is also a scheme.*

Proof. Just note that for an affine open $\operatorname{Spec} A \subseteq X$, $\operatorname{Spec} A \cap U$ can be covered by $D_{\operatorname{Spec} A}(f)$ with some $f \in A$, which are all affine. □

Definition 1.3.3. Let X be a scheme, and $x \in X$ be a point. We define the **residue field** at x to be $\mathcal{O}_{X,x}/\mathfrak{m}_x$, where $\mathcal{O}_{X,x}$ is the stalk of the structure sheaf at x, and $\mathfrak{m}_x$ is its unique maximal ideal.

Lemma 1.3.4. *Let* $\operatorname{Spec} A$ *be an affine scheme. The residue field at a point* $\mathfrak{p} \in \operatorname{Spec} A$ *is just* $\operatorname{Frac} A/\mathfrak{p}$.

The innovative philosophy by Grothendieck suggests that we should consider the properties of morphisms rather than taking effort in the properties of schemes itself. So we will only give several properties of scheme objects here. Then we start to work on morphisms.

Definition 1.3.5. A scheme X is called **quasi-compact**, if its underlying topological space is quasi-compact. It is called **locally Noetherian**, if there exists an affine cover $X = \bigcup_i \operatorname{Spec} A_i$ such that each A_i is a Noetherian ring. It is called **Noetherian**, if it is locally Noetherian and quasi-compact.

Lemma 1.3.6. *A scheme is noetherian if and only if it has a finite affine cover $X = \bigcup_{i=1}^n \operatorname{Spec} A_i$ such that each A_i is a Noetherian ring.*

Lemma 1.3.7. *Let X be a scheme, $X = \bigcup_i \operatorname{Spec} A_i$ be an affine cover. Then*

$$\dim X = \sup_i \dim A_i$$

Definition 1.3.8. A scheme X is called **reduced**, if the stalk $\mathcal{O}_{X,x}$ is a reduced ring for all $x \in X$.

Lemma 1.3.9. *An affine scheme* $\operatorname{Spec} A$ *is reduced if and only if A is a reduced ring.*

Definition 1.3.10. A scheme X is called **integral**, if $\mathcal{O}_X(U)$ is an integral domain for all $U \subseteq X$ affine.

Lemma 1.3.11. *An affine scheme* $\operatorname{Spec} A$ *is integral if and only if A is an integral domain.*

Proposition 1.3.12. *A scheme X is integral if and only if it is irreducible and reduced.*

Proof. Assume that X is integral. If X is reducible, then one finds disjoint opens $U_1, U_2 \subseteq X$. Taking affine opens $\operatorname{Spec} A_1 \subseteq U_1, \operatorname{Spec} A_2 \subseteq U_2$, we conclude that

$$\operatorname{Spec} A_1 \cup \operatorname{Spec} A_2 = \operatorname{Spec} A_1 \coprod \operatorname{Spec} A_2 \cong \operatorname{Spec} A_1 \times A_2$$

is affine, but $A_1 \times A_2$ is not integral, a contradiction. Reducedness is clear, as localization of integral domains is again integral domain and hence reduced.
Now assume that X is irreducible and reduced. Take $U \subseteq X$ affine. Then U is also irreducible and reduced. Hence $Nil(A) = 0$ is a prime ideal, and U is an integral domain. ◻

Definition 1.3.13. A scheme X is called **nonsingular**, or **regular** if it is locally noetherian and the stalk $\mathcal{O}_{X,x}$ is regular for all $x \in X$.

Lemma 1.3.14. *An affine scheme* $\operatorname{Spec} A$ *is nonsingular if and only if* A *is noetherian and regular.*

Like the three definitions above, there are actually two major kind of algebraic properties on schemes: stalk-local properties and affine-local properties.

Definition 1.3.15. A scheme property P is called **stalk-local**, if there is a ring property Q, such that a scheme X has property P if and only if all stalks $\mathcal{O}_{X,x}$ have property Q.

Definition 1.3.16. A scheme property P is called **affine-local**, if for an arbitrary affine scheme $\operatorname{Spec} A$ the following two conditions hold:

1. $\operatorname{Spec} A$ has property P implies that $D(f) = \operatorname{Spec} A[f^{-1}]$ has property P for all $f \in A$.

2. If there exists $f_1, \cdots, f_n \in A$ such that $\bigcup_{i=1}^n D(f_i) = \operatorname{Spec} A$ (i.e. f_i generate the unit ideal in A), and all $D(f_i)$ have property P, then $\operatorname{Spec} A$ has property P.

It follows then immediately from the definition that reducedness and regularity are stalk-local. Integrality is affine-local.

Proposition 1.3.17. *stalks-local properties are affine-local.*

In general the definition of an affine local property is given in the form: X has P if and only if U has P for all $U \subseteq X$ affine. It is however really hard to check properties on each affine open, since we even do not know how many affine opens there are in a scheme. Luckily we have the following theorem which asserts that it will be suffice to just check on one open affine cover.

Theorem 1.3.18 (Affine Communication Lemma)**.** *Let P be an affine-local property. Then X has property P if and only if there exists an affine cover $X = \bigcup_i U_i$ such that each U_i has property P.*

Proof. [3] Tag 01OO. □

Now we can devote ourselves in the long journey of studying the properties of morphisms. One special case is that we want to study all schemes lying over a specific scheme (for instance, $\operatorname{Spec} k$), which motivates the following definition.

Definition 1.3.19. We call $\mathbf{Sch}/S$ the **category of schemes lying over** S. Its objects are tuples (X, f_X) where X is a scheme and f_X is a morphism from X to S, which we call the **structure morphism**. The morphisms g in $\mathbf{Sch}/S$ are scheme morphisms $X \to Y$ making the following diagram commute:

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "f_X"'] & & Y \arrow[dl, "f_Y"] \\ & S & \end{tikzcd}$$

Theorem 1.3.20. *The fibre product $Y \times_X Z$ exists for all X, Y, Z in the category of schemes. If $X = \operatorname{Spec} A, Y = \operatorname{Spec} B, Z = \operatorname{Spec} C$, then $Y \times_X Z = \operatorname{Spec} B \otimes_A C$.*

Proof. [1] Theorem II.3.3. The basic idea is to glue $\operatorname{Spec} B \otimes_A C$ for affine opens $\operatorname{Spec} B, \operatorname{Spec} C$ in Y, Z mapping to $\operatorname{Spec} A$ in X. $\square$

Remark 1.3.21. It is in general not true that $|Y \times_X Z| = |Y| \times_{|X|} |Z|$. Consider the product of two affine line $\mathbb{A}^1_k \times_{\operatorname{Spec} k} \mathbb{A}^1_k$. It is just $\operatorname{Spec} k[X] \otimes_k k[X] = \operatorname{Spec} k[X, Y]$. But $k[X, Y]$ contains apparently more prime ideals than $|\operatorname{Spec} k[X]| \times |\operatorname{Spec} k[X]|$.

The category of schemes has thus good properties ensuring us to pull back schemes along morphisms between base schemes.

Definition 1.3.22. A property P of scheme morphisms is said to satisfy
1.**COMP**, if it is stable under compositions.
2.**BC**, if it is stable under base change, i.e. given $f : Y \to X$ with property P satisfying BC, then for any cartesian square

$$\begin{tikzcd} Z \times_X Y \arrow[r] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ Z \arrow[r] & X \end{tikzcd}$$

the morphism g also has property P.
3.**LOCT**, if it can be checked locally on target, i.e. $f : Y \to X$ has property P, if for one open covering $X = \bigcup_i U_i$, all of $f|_{f^{-1}(U_i)}$ have P.
3.**LOCS**, if it can be checked locally on source, i.e. $f : Y \to X$ has property P, if for one open covering $Y = \bigcup_i V_i$, all of $f|_{V_i}$ has P.

The COMP is clearly of great importance. And in algebraic geometry it is a general technique to pull back a scheme onto a base scheme whose structure is well understood, for example an algebraically closed field, so we would like to keep the property of morphisms after base change. Therefore a sensible definition of property should always ensure BC.

Remark 1.3.23. Without further mentions, all the properties of morphisms defined in this notes actually satisfy COMP and BC.

Definition 1.3.24. A morphism of schemes is called a **quasi-compact morphism**, if the preimage of any quasi-compact open is quasi-compact.

Definition 1.3.25. A morphism $f : Y \to X$ is called **locally of finite type**, if for all affine $U \subseteq X$, $f_*\mathcal{O}_Y(U)$ is a $\mathcal{O}_X(U)$-Algebra of finite type. It is called **of finite type**, if f is locally of finite type and quasi-compact.

Lemma 1.3.26. *A morphism $f : Y \to X$ being locally of finite type is affine-local on X, thus f is locally of finite type if and only if there there exists an affine cover $X = \bigcup_i U_i$, such that $f_*\mathcal{O}_Y(U_i)$ is a $\mathcal{O}_X(U_i)$-Algebra of finite type for each i.*

Definition 1.3.27. Let X, Y be schemes, $f : Y \to X$ be a morphism.
1. f is called an **open immmersion**, if f is an open embedding on the underlying topological spaces and $f^\sharp : f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ is an isomorphism of sheaves.
2. f is called a **closed immersion**, if f is a closed embedding on the underlying topological spaces and $f_\flat : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective.
3. f is called a **locally closed immersion**, if it can be factored as as $j \circ i$, where i is a closed immersion and j is an open immersion.

4. Y is called an **open (resp. closed) subscheme** of X, if one has an open (resp. closed) immersion $Y \to X$.

Lemma 1.3.28. *Let X, Y be schemes, $f : Y \to X$ be a morphism.*
1. If f is an open immersion, then $Y \cong f(Y)$.
2. If f is a closed immersion, and $X = \operatorname{Spec} A$ is affine, then $Y = \operatorname{Spec} A/I$ for some ideal I.

Lemma 1.3.29. *Let A be a ring, $I \subseteq A$ be an ideal, $\phi : A \to A/I$ be the canonical quotient map. Then the induced scheme morphism $\operatorname{Spec} f : \operatorname{Spec} A/I \to \operatorname{Spec} A$ is a closed immersion with topological image $V(I)$. We denote with $V(I)$ since now the closed subscheme $\operatorname{Spec} A/I$.*

Remark 1.3.30. We have according to the two lemmata a one-to-one correspondence between closed subschemes of $\operatorname{Spec} A$ and ideals of A. Also note that $V(I)$ and $V(I^2)$ have the same underlying topological space, but different structure sheaves.

Proposition 1.3.31. *Let $U \subseteq X$ be an open subscheme, $f : Y \to X$ be a morphism. Then the open subscheme $f^{-1}(U)$ in Y fits into a cartesian square:*

$$\begin{tikzcd}
f^{-1}(U) \arrow[r, hook] \arrow[d, "f|_{f^{-1}(U)}"'] & Y \arrow[d, "f"] \\
U \arrow[r, hook] & X
\end{tikzcd}$$

Definition 1.3.32. Extending the notion of preimage, we define the preimage of a subscheme $Z \hookrightarrow X$ under $f : Y \to X$ to be $Z \times_X Y$. Similarly, we define the **scheme theoretic fibre** of f at a point $x \in X$ to be $\operatorname{Spec} k(x) \times_X Y$, where $k(x)$ is the residue field at x.

Proposition 1.3.33. *If $f : Y \to X$ is a closed immersion, then the preimage of an affine open $\operatorname{Spec} A \subseteq X$ is $\operatorname{Spec} A/I$ for some ideal I.*

More generally, the functor $U \mapsto \ker(f_\flat(U) : \mathcal{O}_X(U) \to f_*\mathcal{O}_Y(U))$ is a sheaf, called the sheaf of ideals corresponding to f. It will be thoroughly discussed in the next chapter.

Definition 1.3.34. A morphism $f : Y \to X$ is called an **affine morphism** if for all affine open $U \subseteq X$ the preimage $f^{-1}(U)$ is also affine.

Lemma 1.3.35. *A morphism $f : Y \to X$ being affine is an affine-local property on X, thus f is affine if and only if there exists an affine cover $X = \bigcup_i U_i$ such that $f^{-1}(U_i)$ is affine for each i.*

Corollary 1.3.36. *Closed immersions are affine.*

Definition 1.3.37. A morphism $f : Y \to X$ is called a **finite morphism**, if f is affine and for all affine open $U \subseteq X$, $\mathcal{O}_Y(f^{-1}(U))$ is a finite $\mathcal{O}_X(U)$-module.

Lemma 1.3.38. *A morphism $f : Y \to X$ being finite is an affine-local property on X, thus f is finite if and only if there exists an affine cover $X = \bigcup_i U_i$ such that $\mathcal{O}_Y(f^{-1}(U_i))$ is a finite $\mathcal{O}_X(U_i)$-module for each i.*

Definition 1.3.39. Let $f : Y \to X$ be a morphism of schemes. The **diagonal morphism** $\Delta_f : Y \to Y \times_X Y$ is defined through the following diagram:

$$\begin{tikzcd}
Y \arrow[dr, dashed, "\Delta_f"] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "id"'] & & \\
& Y \times_X Y \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\
& Y \arrow[r, "f"] & X
\end{tikzcd}$$

Proposition 1.3.40. *The diagonal morphism is a locally closed immersion.*

Proof. Just note that we can take an affine cover $Y = \bigcup_i U_i$, then $\bigcup_i U_i \times_X U_i$ is an open subscheme of $Y \times_X Y$. The diagonal lies completely in $\bigcup_i U_i \times_X U_i$, and for each $U_i \times_X U_i$, the preimage of diagonal is U_i. Moreover, $U_i \to U_i \times_X U_i$ is a closed immersion, corresponding to the multiplication map $B \otimes_A B \to B, b_1 \otimes b_2 \mapsto b_1 b_2$. The result follows. $\square$

Definition 1.3.41. A morphism $f : Y \to X$ is called a **quasi-separated morphism**, if the corresponding diagonal morphism Δ_f is quasi-compact. f is called **separated**, if Δ_f is a closed immersion.

Remark 1.3.42. The motivation of separated morphism is that in the category of topological spaces, a space X is Hausdorff if and only if the diagonal $\Delta : X \to X \times X$ is a closed immersion.

Proposition 1.3.43. *Let X be a separated scheme over the ring $\mathbb{Z}$, then the intersection $U_1 \cap U_2$ of two affine opens U_1, U_2 is again affine.*

Proof. We first note that $U_1 \cap U_2 \cong U_1 \times_X U_2$, by showing $U_1 \cap U_2$ satisfies the universal property of fibre product. Then we use the magic diagram on the left:

$$\begin{tikzcd} X_1 \times_Y X_2 \arrow[r] \arrow[d] & X_1 \times_Z X_2 \arrow[d] \\ Y \arrow[r, "\Delta"] & Y \times_Z Y \end{tikzcd} \qquad \begin{tikzcd} U_1 \times_X U_2 \arrow[r] \arrow[d] & U_1 \times_{\mathbb{Z}} U_2 \arrow[d] \\ X \arrow[r, "\Delta"] & X \times_{\mathbb{Z}} X \end{tikzcd}$$

By substituing the notations, we get the right Cartesian diagram. Since the diagonal morphism is closed immersion, the arrow above is by base change also a closed immersion, in particular affine. But $U_1 \times_{\mathbb{Z}} U_2$ is affine, so $U_1 \cap U_2$ is affine. $\square$

Just like separatedness for Hausdorffness, we have an analogue of compact Hausdorfness in algebraic geometry, called properness.

Definition 1.3.44. A morphism $f : Y \to X$ is called a **proper morphism** if it is separated, of finite type and universally closed (i.e. the map on the underlying topological spaces is closed, and stable under base change).

Separatedness and universal closedness are in general hard to check. Luckily we have a nice criterion for them.

Theorem 1.3.45 (Valuative Criterion)**.** *Let $f : Y \to X$ be a morphism of schemes, V a valuation ring with fraction field K, and given an (not necessarily cartesian) square diagram on the left*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & Y \arrow[d] \\ \operatorname{Spec} V \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & Y \arrow[d] \\ \operatorname{Spec} V \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

1. *Assume f is quasi-separated. Then f is separated if and only if there exists at most one lift* $\operatorname{Spec} V \to Y$ *for each V and square.*

2. *Assume f is quasi-compact. Then f is universally closed if and only if there exists at least one lift* $\operatorname{Spec} V \to Y$ *for each* V *and square.*

3. *Assume f is quasi-separated and of finite type. Then f is proper if and only if there exists a unique lift* $\operatorname{Spec} V \to Y$ *for each* V *and square.*

Proof. [3] Tag 01KA, Tag 01KY. □

Remark 1.3.46. The intuition of valuative criterion is that we consider the spectrum of a discrete valuation ring $\operatorname{Spec} V$, which has two points, endowed with Sierpinski topology. The non-separatedness of the affine line with doubled origin reflects as that the two lines has one shared generic point, but the two origins are not separable in the "Hausdorff sense". We can then map the open point in $\operatorname{Spec} V$ to the generic point, and the closed point to any of the origins. Then we may conclude with the valuative criterion that the affine line with doubled origin is not separated.
For an arbitrary quasi-separated scheme, roughly (i.e. not rigorously) speaking, the valuative criterion helps us check whether the scheme contains an affine line with doubled origin as a subscheme, and hence whether it is separated or not.

Definition 1.3.47. Let k be a field. A **variety** X over k is a scheme over k that is integral, separated and of finite type.

We finally state the cancellation theorem, which can almost always be applied when the structure morphism is separated.

Theorem 1.3.48 (Cancellation Theorem)**.** *Let X, Y be schemes over S and $f : X \to Y$ a morphism over S.*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\phi_X"'] & & Y \arrow[dl, "\phi_Y"] \\ & S & \end{tikzcd}$$

Suppose that ϕ_X has property P satisfying COMP and BC, and the diagonal morphism $\Delta_{Y/S}$ corresponding to ϕ_Y has property P, then f has property P.

Proof. Again we use the magic diagram and substituting the elements properly we get the graph diagram on the right:

$$\begin{tikzcd} X_1 \times_Y X_2 \arrow[r] \arrow[d] & X_1 \times_Z X_2 \arrow[d] \\ Y \arrow[r, "\Delta"] & Y \times_Z Y \end{tikzcd} \qquad \begin{tikzcd} X \times_Y Y \cong X \arrow[r, "\Gamma"'] \arrow[d, "f"] & X \times_Z Y \arrow[d, "f\times \mathrm{Id}"'] \\ Y \arrow[r, "\Delta"] & Y \times_Z Y \end{tikzcd}$$

So the graph morphism Γ has property P. Again by the fibre product diagram

$$\begin{tikzcd} X \times_S Y \arrow[r] \arrow[d, "p_Y"] & X \arrow[d, "\phi_X"'] \\ Y \arrow[r] & S \end{tikzcd}$$

the projection p_Y has property P. Therefore $f = p_Y \circ \Gamma$ has property P. $\square$

1.4 Construction of Proj and Projective Spaces

In this section we discuss a very important construction, namely the homogeneous spectrum Proj of a graded ring. The motivation is that for an algebraically closed field k, the points $[a_0 : \cdots : a_n]$ are in one-to-one correspondence to ideals of the form $(a_iX_j - a_jX_i)_{i,j}$ in $k[X_0, \dots, X_n]$, which are maximal among all homogeneous ideals. So it is natural to consider the set of all homogeneous prime ideals in a graded ring, which can be seen as the generalization of the projective space. Like Spec we first define $\operatorname{Proj} S$ as a set, then we give it the Zariski topology, and finally make it a scheme with a structure sheaf.

Remark 1.4.1. Without further notation, a graded ring $S = \bigoplus_{n\geq 0} S_n$ is always $\mathbb{N}$-graded and generated by S_1 as an S_0-algebra. We write then $S_+ := \bigoplus_{n\geq 1} S_n$ for the **irrelevant ideal**. There are more general definitions for graded rings not generated in degree one, but in algebraic geometry we concern ourselves almost only with the case for a polynomial ring over a field or its quotient ring, and the constructions will also be nicer and more clean.

Definition 1.4.2. Let S be a graded ring. We define the **homogeneous spectrum** $\operatorname{Proj} S$ to be the set of all homogeneous prime ideals except those containing the irrelevant ideal:

$$\operatorname{Proj} S := \{\mathfrak{p} \subset S | \mathfrak{p} \text{ homogeneous prime}, \mathfrak{p} \not\supseteq S_+\}$$

If $I \subseteq S$ is a homogeneous ideal, we define the **homogeneous vanishing locus** $V_+(I)$ in $\operatorname{Proj} S$ to be

$$V_+(I) := \{\mathfrak{p} \in \operatorname{Proj} S | I \subseteq \mathfrak{p}\}$$

Proposition 1.4.3. *Just like the affine case, we have*

$$\bigcap_i V_+(I_i) = V_+\left(\sum_i I_i\right).$$
$$V_+(I_1) \cup V_+(I_2) = V_+(I_1 \cap I_2).$$

Definition 1.4.4. The homogeneous vanishing loci of homogeneous ideals as closed sets define a topology on $\operatorname{Proj} S$. We call it **Zariski Topology**.

Remark 1.4.5. $\operatorname{Proj} S$ can be naturally thought as a subset of $\operatorname{Spec} S$. One can also identify $V_+(I) = V(I) \cap \operatorname{Proj} S$. Thus $\operatorname{Proj} S$ has actually the subspace topology.

Unfortunately the Proj does not have a functoriality like Spec. Indeed, if $f : S \to T$ is a homogeneous ring homomorphism, then the preimage $f^{-1}(\mathfrak{p})$ for $\mathfrak{p} \in \operatorname{Proj} T$ may still contain the irrelevant ideal S_+. However, we see that this bad case does not happen if f is surjective, thus f gives a closed embedding $\operatorname{Proj} S/I \hookrightarrow \operatorname{Proj} S$.

Definition 1.4.6. Let $f \in S$ be a homogeneous element.We call $D_+(f) := \operatorname{Proj} S \backslash V_+(f)$ a **homogeneous principal open**.

Lemma 1.4.7. *The set of all homogeneous principal opens in* $\operatorname{Proj} S$ *forms a basis of Zarisiki Topology and is stable under intersections. In particular,* $D_+(f) \cap D_+(g) = D_+(fg)$

Proposition 1.4.8. *The homogeneous principal open* $D_+(f)$ *in* $\operatorname{Proj} S$ *is homeomorphic to* $|\operatorname{Spec} S_{(f)}|$*, where* $S_{(f)}$ *means the homogeneous localization at* f*. The closed locus* $V_+(I)$ *is homeomorphic to* $\operatorname{Proj} A/I$*.*

Theorem 1.4.9 (Homogeneous Nullstellensatz)**.** *Let S be a graded ring. Then there is a bijection between closed subsets of* $\operatorname{Proj} S$ *and homogeneous radical ideals in S_+, by sending $V \subseteq \operatorname{Spec} A$ to $I_+(V) := \bigcap_{\mathfrak{p}\in V} \mathfrak{p}$, and $I \subseteq A$ to $V_+(I)$.*

Proof. It follows directly from the general Nullstellensatz 1.2.1 □

Remark 1.4.10. Note that we need an extra condition $I \subset S_+$. A counterexample is $\operatorname{Proj} \mathbb{Z}[X]$. The ideals (2) and $(2X)$ cut out the same locus since $V_+(2X) = V_+(2) \cup V_+(X) = V_+(2) \cup \emptyset = V_+(2)$, and they are both radical.

Proposition 1.4.11. *Let S be a graded ring, $\mathfrak{p}$ be a homogeneous prime ideal. Then the closed subset $V_+(\mathfrak{p})$ is irreducible.*

Remark 1.4.12. Note that unlike the affine case the converse does not hold. The irreducibility of $V \subset \operatorname{Proj} S$ does not imply that $I_+(V)$ is prime. Just consider $\operatorname{Proj} \mathbb{Z}[X, Y]$. $V_+(2X)$ is just a singleton but $I_+(V_+(2X)) = (2X)$ is not prime.

Lemma 1.4.13. *If S is a Noetherian graded ring, then* $\operatorname{Proj} S$ *is a Noetherian topological space.*

Now we start to give the homogeneous spectrum a scheme structure. For this we need one small lemma.

Lemma 1.4.14. *Let S be a graded ring, $f, g \in S$ homogeneous. Then the three localizations $(S_{(f)})[(g^{deg\, f}/f^{deg\, g})^{-1}]$, $(S_{(g)})[(g^{deg\, f}/f^{deg\, g})^{-1}]$ and $S_{(fg)}$ are canonically isomorphic. In particular, there underlying topological spaces of spectra are all homeomorphic to $D_+(fg) \subseteq \operatorname{Proj} S$.*

Remark 1.4.15. If f, g are homogeneous in degree 1, we will have a nice form $S_{(f)}[(g/f)^{-1}]$ and $S_{(g)}[(f/g)^{-1}]$.

Definition 1.4.16. Let S be a graded ring. We endow every homogeneous principal open $D_+(f) \cong |\operatorname{Spec} S_{(f)}|$ with the structure sheaf $\mathcal{O}_{\operatorname{Spec} S_{(f)}}$. This definition agrees on overlaps, by 1.4.14. Thus $(\operatorname{Proj} S, \mathcal{O}_{\operatorname{Proj} S})$ becomes a scheme.

Remark 1.4.17. Since now we write $\operatorname{Proj} S$ for the scheme in the definition, and $|\operatorname{Proj} S|$ for the underlying topological space.

The construction Proj is not completely functorial, because even if we have a graded ring homomorphism $A \to B$ and a homogeneous prime ideal $\mathfrak{p}$ in B not contained in B_+, its preimage can still contain A_+. We give however some special cases where a graded ring homomorphism indeed extends to a morphism between two Proj.

Lemma 1.4.18. *Let $f : A \to B$ be a surjective graded ring homomorphism. Then it induces a closed immersion of schemes* $\operatorname{Proj} f : \operatorname{Proj} B \to \operatorname{Proj} A$.

Lemma 1.4.19. *Let $f : A \to B_0, g : A \to A'$ be two ring homomorphisms. Then the canonical map* $\mathrm{Id} \otimes 1 : B \to B \otimes_A A'$ *induces a morphism of schemes* $\operatorname{Proj} \mathrm{Id} \otimes 1 : \operatorname{Proj} B \otimes_A A' \to \operatorname{Proj} B$.

Remark 1.4.20. In the two situations above, we have actually $f^*\mathcal{O}_Y(n) \cong \mathcal{O}_X(n)$. We will come back to the definition of $\mathcal{O}(n)$ later in Chapter 2.

Definition 1.4.21. The **projective n-space** over a ring A $\mathbb{P}^n_A$ is defined as $\operatorname{Proj} A[X_0, \ldots, X_n]$, where all elements in A have degree 0, and all X_i's have degree 1.

Remark 1.4.22. Note that by construction one has a natural structure morphism $\mathbb{P}^n_A \to \operatorname{Spec} A$.

Proposition 1.4.23. *Let A be a ring. We have $\Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(d)) = A[X_0, \ldots, X_n]_d$, i.e. the degree d part of the polynomial ring.*

In the world of projective spaces we lose a lot of good properties in affine cases, but at this cost we gain that the projective n-space is proper over A.

Proposition 1.4.24. *Let A be a ring, then the natural structure morphism* $\mathbb{P}^n_A \to \operatorname{Spec} A$ *is proper.*

Proof. The proof needs the theory of invertible sheaves, which is discussed in Chapter 2. First note that $\mathbb{P}^n_A$ is of finite type and quasi-separated over A. Take a (not necessarily Cartesian) diagram:

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r, "f"] \arrow[d] & \mathbb{P}^n_A \arrow[d] \\
\operatorname{Spec} V \arrow[r] \arrow[ur, dashed, "g"] & \operatorname{Spec} A
\end{tikzcd}$$

where V is a valuation ring and K its fraction field. f corresponds to a line bundle on $\operatorname{Spec} K$ and $n+1$ global sections. But a line bundle on $\operatorname{Spec} K$ is always isomorphic to K itself. Therefore we see f corresponds to $n+1$ elements $a_0, \dots, a_n$ in K. Take a_i with the smallest valuation, then $a_0/a_i, \dots, a_n/a_i$ defines the same map f since K is isomorphic to itself as K-vector space by multiplying with a_i. Now a_j/a_i are in V for all j. So they define a map g to $\mathbb{P}^n$. By construction, the diagram above commutes. This shows the existence part.
Now assume that two tuples $a_0, \dots, a_n$, $b_0, \dots, b_n$ in V define g_1, g_2 to $\mathbb{P}^n$, but coincide on $\operatorname{Spec} K$. This implies there exists $\lambda \in K^*$ such that $a_j = \lambda b_j$. But either λ or λ^{-1} is in V, so by multiplying with either λ or λ^{-1} we make the two tuples isomorphic. Hence $g_1 = g_2$. □

Lemma 1.4.25. $\dim \mathbb{P}^n_A = \dim A + n$. *In particular, if k is a field, then* $\dim \mathbb{P}^n_k = n$.

Definition 1.4.26. We call a scheme X projective over a ring A, if the structure morphism $f : X \to \operatorname{Spec} A$ factors through a closed immersion to a projective n-space over A.

$$\begin{tikzcd} X \arrow[rr, hook, "\iota"] \arrow[dr, "f"'] & & \mathbb{P}^n_A \arrow[dl] \\ & \operatorname{Spec} A & \end{tikzcd}$$

The study of projective varieties is of great importance in algebraic geometry. We will discuss more details in the later chapters with more powerful tools such as ample line bundles and cohomology.

Chapter 2

Modules on Schemes

Since we use schemes to extend the notion of a ring, a natural idea is to study the modules on schemes, which we will spend one whole chapter to discuss.

2.1 Sheaves of Modules and Quasi-coherent Sheaves

Definition 2.1.1. Let X be a scheme. A **sheaf of $\mathcal{O}_X$-modules** is a sheaf of abelian groups $\mathcal{M}$, together with a morphism of sheaves (scalar multiplication) $\mathcal{O}_X \times \mathcal{M} \to \mathcal{M}$, such that $\mathcal{M}(U)$ is a $\mathcal{O}_X(U)$-module via the map $\mathcal{O}_X(U) \times \mathcal{M}(U) \to \mathcal{M}(U)$. A morphism of $\mathcal{O}_X$-modules is just a morphism of sheaves respecting the module structure, i.e. the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}_X \times \mathcal{M} & \xrightarrow[\mathrm{Id}\times f]{} & \mathcal{O}_X \times \mathcal{N} \\ \downarrow & & \downarrow \\ \mathcal{M} & \xrightarrow{f} & \mathcal{N} \end{array}$$

Remark 2.1.2. Note that every $\mathcal{M}(U)$ has an $\mathcal{O}_X(U)$-module structure, and every stalk $\mathcal{M}_x$ has an $\mathcal{O}_{X,x}$-module structure.

Proposition 2.1.3. *Let X be a scheme. Then all $\mathcal{O}_X$-modules and their morphisms form an abelian category.*

Definition 2.1.4. Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_X$-modules on a scheme X. The **support** of $\mathcal{M}$ is defined as

$$\mathrm{supp}\mathcal{F} := \{x \in X | \mathcal{M}_x \neq 0\}.$$

Definition 2.1.5. Let X be a scheme, $\mathcal{M}, \mathcal{N}$ be sheaves of $\mathcal{O}_X$-modules. We define $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$, called the **tensor product** of $\mathcal{M}, \mathcal{N}$, to be the sheafification of the presheaf $U \mapsto \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{N}(U)$.

Lemma 2.1.6. *We have for all points $x \in X$ a canonical isomorphism:*

$$(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N})_x \cong \mathcal{M}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{N}_x$$

Definition 2.1.7. Let X be a scheme, $\mathcal{M}, \mathcal{N}$ be sheaves of $\mathcal{O}_X$-modules. We define the **internal Hom sheaf** from $\mathcal{M}$ to $\mathcal{N}$ to be $\mathfrak{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})(U) := \mathrm{Hom}_{\mathcal{O}_U-\mathrm{Mod}}(\mathcal{M}|_U, \mathcal{N}|_U)$. This is a sheaf of $\mathcal{O}_X$-modules.

Lemma 2.1.8. *We have $\mathfrak{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})|_U = \mathfrak{Hom}_{\mathcal{O}_U}(\mathcal{M}|_U, \mathcal{N}|_U)$.*

Proposition 2.1.9 (Tensor-Hom Adjunction)**.** *Let X be a scheme, $\mathcal{M}, \mathcal{N}, \mathcal{L}$ be sheaves of $\mathcal{O}_X$-modules. There is a canonical isomorphism*

$$\mathfrak{Hom}_{\mathcal{O}_X}(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}, \mathcal{L}) \cong \mathfrak{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathfrak{Hom}_{\mathcal{O}_X}(\mathcal{N}, \mathcal{L}))$$

functorial in $\mathcal{M}, \mathcal{N}, \mathcal{L}$. In particular, the functors $- \otimes \mathcal{N}$ and $\mathfrak{Hom}(\mathcal{N}, -)$ are adjoints.

Definition 2.1.10. Let X, Y be schemes, $f : X \to Y$ a morphism of schemes, $\mathcal{M}$ a sheaf of $\mathcal{O}_X$-modules, $\mathcal{N}$ a sheaf of $\mathcal{O}_Y$-module. We endow the **pushforward** $f_*\mathcal{M}$ with the inheriting $\mathcal{O}_Y$-module structure and consider it as a sheaf of $\mathcal{O}_Y$-module. We define the **pullback** $f^*\mathcal{M}$ to be the sheaf of $\mathcal{O}_X$-modules $\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_X} \mathcal{M}$.

Lemma 2.1.11. *Let X, Y be schemes, $f : X \to Y$ a morphism of schemes, then $f^*\mathcal{O}_Y = \mathcal{O}_X$.*

Proposition 2.1.12 (Pullback-Pushforward Adjunction)**.** *Let X, Y be schemes, $f : X \to Y$ a morphism of schemes, $\mathcal{M}$ a sheaf of $\mathcal{O}_X$-modules, $\mathcal{N}$ a sheaf of $\mathcal{O}_Y$-modules. Then there is an isomorphism*

$$\mathrm{Hom}_{\mathcal{O}_X-\mathrm{Mod}}(f^*\mathcal{N}, \mathcal{M}) \cong \mathrm{Hom}_{\mathcal{O}_Y-\mathrm{Mod}}(\mathcal{N}, f_*\mathcal{M})$$

functorial in $\mathcal{M}, \mathcal{N}$.

Proof. By 1.1.13 we have an adjunction of sheaves

$$\mathrm{Hom}_{f^{-1}\mathcal{O}_Y-\mathrm{Mod}}(f^{-1}\mathcal{N},\mathcal{M}) \cong \mathrm{Hom}_{\mathcal{O}_Y-\mathrm{Mod}}(\mathcal{N}, f_*\mathcal{M})$$

and we have always the bijection

$$\mathrm{Hom}_{f^{-1}\mathcal{O}_Y-\mathrm{Mod}}(f^{-1}\mathcal{N},\mathcal{M}) \cong \mathrm{Hom}_{\mathcal{O}_X-\mathrm{Mod}}(f^{-1}\mathcal{N}\otimes\mathcal{O}_X,\mathcal{M})$$

The result follows. □

Definition 2.1.13. Let X be a scheme, $\mathcal{M}$ a sheaf of $\mathcal{O}_X$-modules. We call $\mathcal{M}$ is
1. **free**, if there is an isomorphism $\mathcal{M} \cong \mathcal{O}_X^{\oplus I}$ for some index set I.
2. **locally free**, if there is an open cover $X = \bigcup_i U_i$, such that $\mathcal{M}|_{U_i}$ is free on U_i for all i. We also say that $\mathcal{M}$ is a **vector bundle** if it is locally free of finite rank.
3. **locally projective**, if for each affine $\operatorname{Spec} A \subseteq X$, $\mathcal{M}(\operatorname{Spec} A)$ is a projective A-module.
4. **globally generated**, if there is a surjection $\mathcal{O}_X^{\oplus I} \to \mathcal{M}$.
5. **of finite type**, if there exists an open cover $X = \bigcup_i U_i$ such that for each i there exists an integer n_i, and a surjection $\mathcal{O}_{U_i}^{\oplus n_i} \to \mathcal{M}|_{U_i}$.
6. **of finite presentation**, if there exists an open cover $X = \bigcup_i U_i$ such that for each i there exists two integers m_i, n_i, and an exact sequence:

$$\mathcal{O}_{U_i}^{\oplus m_i} \to \mathcal{O}_{U_i}^{\oplus n_i} \to \mathcal{M}|_{U_i} \to 0$$

i.e. the kernel of the generating morphism in 4. is again of finite type.

Remark 2.1.14. Note that a morphism $f : \mathcal{O}_Y \to \mathcal{O}_X$ locally of finite type does not imply $f_*\mathcal{O}_Y$ is an $\mathcal{O}_X$-module of finite type. Indeed, one definition means locally finitely generated as an algebra and the other means locally finitely generated as a module. Meanwhile, f is finite is equivalent to $f_*\mathcal{O}_Y$ is an $\mathcal{O}_X$-module of finite type.

Lemma 2.1.15. *All the properties in the definition above are preserved under pullback.*

Lemma 2.1.16. *Local projectiveness is an affine-local property.*

Proof. [3] Tag 05JQ. □

Definition 2.1.17. Let X be a scheme $\mathcal{M}$ a vector bundle on X. Let $s \in \Gamma(X, \mathcal{M})$ be a global section. We define the **vanishing locus** $V(s)$ in the following way: Pick an affine cover $X = \bigcup_i \operatorname{Spec} A_i$. Then $s|_{\operatorname{Spec} A_i}$ can be viewed as an element of $\widetilde{A_i^{\oplus n}}$. Find a representative $(f_1, \dots, f_n)$ of s, where $f_j \in A_i$. Define $V(s)$ on $\operatorname{Spec} A_i$ to be $V(f_1, \dots, f_n)$. This definition glues well on overlaps. Furthermore we define $D(s) := X \backslash V(s)$.

Now a natural question: For an affine scheme $\operatorname{Spec} A$, how are the $\mathcal{O}_{\operatorname{Spec} A}$-modules and A-modules related?

Proposition 2.1.18. *Let A be a ring, M be an A-module. We associate every principal open $D(f) \in \operatorname{Spec} A$ with the $A[f^{-1}]$-module $M[f^{-1}]$, and define the restriction map $\operatorname{res}^{D(f)}_{D(g)}$ to be the canonical localization $M[f^{-1}] \to M[g^{-1}]$. These data define a sheaf on the principal opens of* Spec *A, and by 1.1.4 extend to a sheaf of abelian groups on* Spec *A, and it has a natural O_x-module structure. We call it **the sheaf of $\mathcal{O}_{\operatorname{Spec} A}$-modules asscoiated to** M, denoted $\widetilde{M}$.*

Definition 2.1.19. Let X be a scheme, a **quasi-coherent sheaf** on X is a sheaf of $\mathcal{O}_X$-modules $\mathcal{M}$ such that for every affine open $\operatorname{Spec} A \in X$, $\mathcal{M}|_{\operatorname{Spec} A} \cong \widetilde{M}$ for some A-module M. If X is furthermore Noetherian and each M is a finite A-module, $\mathcal{M}$ is called a **coherent sheaf**.

Lemma 2.1.20. *The property of a sheaf of $\mathcal{O}_X$-modules being quasi-coherent is affine-local. In particular, $\mathcal{M}$ over* Spec *A is quasi-coherent if and only if $\mathcal{M} \cong \widetilde{\Gamma(\operatorname{Spec} A, \mathcal{M})}$.*

Lemma 2.1.21. *Let A be a ring, M be an A-module. Then there is a canonical isomorphism $(\widetilde{M})_{\mathfrak{p}} \cong M_{\mathfrak{p}}$, where $(\widetilde{M})_{\mathfrak{p}}$ is the stalk of $\widetilde{M}$ at $\mathfrak{p} \in \operatorname{Spec} A$.*

Lemma 2.1.22. *Let A be a ring, M, N be A-modules, $f : M \to N$ a module homomorphism. Then f induces a morphism of $\mathcal{O}_{\operatorname{Spec} A}$-modules $\widetilde{M} \to \widetilde{N}$.*

Proposition 2.1.23. *Let A be a ring, M an A-module, $\mathcal{N}$ a sheaf of $\mathcal{O}_{\operatorname{Spec} A}$-modules. There is actually an adjunction*

$$\operatorname{Hom}_{\mathcal{O}_X-\mathrm{Mod}}(\widetilde{M}, \mathcal{N}) \cong \operatorname{Hom}_{A-\mathrm{Mod}}(M, \Gamma(\operatorname{Spec} A, \mathcal{N}))$$

Corollary 2.1.24. *Let A be a ring. Then the functor sending an A-module M to the sheaf of $\mathcal{O}_{\operatorname{Spec} A}$-modules $\widetilde{M}$ is an equivalence between the category of A-modules and the category of quasi-coherent sheaves on* $\operatorname{Spec} A$.

Lemma 2.1.25. *Let X be a scheme, $\mathcal{M}, \mathcal{N}$ be two quasi-coherent sheaves. Then on each affine $\operatorname{Spec} A \subseteq X$ where $\mathcal{M}|_{\operatorname{Spec} A} = \widetilde{M}, \mathcal{N}|_{\operatorname{Spec} A} = \widetilde{N}$ we have $\mathfrak{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})(U) \cong \operatorname{Hom}_{A-\mathrm{Mod}}(M, N)$.*

Remark 2.1.26. Despite the lemma, $\mathfrak{Hom}(\mathcal{M}, \mathcal{N})$ need not to be quasi-coherent even if $\mathcal{M}, \mathcal{N}$ are quasi-coherent.

Lemma 2.1.27. *Let X be a scheme, $\mathcal{M}, \mathcal{N}$ be two quasi-coherent sheaves. Then on each affine $\operatorname{Spec} A \subseteq X$ where $\mathcal{M}|_{\operatorname{Spec} A} = \widetilde{M}, \mathcal{N}|_{\operatorname{Spec} A} = \widetilde{N}$ we have $(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N})|_{\operatorname{Spec} A} \cong \widetilde{\mathcal{M} \otimes_A \mathcal{N}}$. In particular, if $f : Y \to X$ is a morphism of schemes, $\mathcal{M}$ a quasi-coherent sheaf on X and $\operatorname{Spec} B \subseteq Y$ maps into $\operatorname{Spec} A \subseteq X$, and denote $\mathcal{M}|_{\operatorname{Spec} A} = \widetilde{M}$. then $f^*\mathcal{M}|_{\operatorname{Spec} B} \cong \widetilde{M \otimes_B A}$.*

Proposition 2.1.28. *Let X be a scheme. The category of quasi-coherent sheaves on X is an abelian category.*

Definition 2.1.29. Let X be a scheme. A **sheaf of ideals** $\mathcal{I}$ on X is a quasi-coherent subsheaf of $\mathcal{O}_X$. The **closed subscheme associated to** $\mathcal{I}$ is the closed subscheme with the underlying topological space $\operatorname{supp}(\mathcal{O}_X/\mathcal{I})$, and the structure sheaf $f^{-1}(\mathcal{O}_X/\mathcal{I})$, where $f : \operatorname{supp}(\mathcal{O}_X/\mathcal{I}) \hookrightarrow X$ is the inclusion of topological spaces.

Proposition 2.1.30. *Let X be a scheme, $\operatorname{Spec} A \subseteq X$ be an affine open, $\mathcal{I}$ a sheaf of ideals on X and Y the associated closed subscheme. Let I be an ideal in A such that $\mathcal{I}|_{\operatorname{Spec} A} = \widetilde{I}$. Then we get a Cartesian diagram*

$$\begin{tikzcd}
\operatorname{Spec}(A/I) \arrow[r, hook] \arrow[d, hook] & Y \arrow[d, hook] \\
\operatorname{Spec} A \arrow[r, hook] & X
\end{tikzcd}$$

where the horizontal arrows are open immersions and the vertical arrows are closed immersions. In particular, closed immersions are affine morphisms.

Proposition 2.1.31. *Let X be a scheme. There is a one-to-one correspondence between closed subschemes of X and ideal sheaves on X, by sending an ideal sheaf to its associated closed subscheme, and sending a closed subscheme $Y \hookrightarrow X$ to* $\ker(\mathcal{O}_X \to f_*\mathcal{O}_Y)$.

2.2 Qcoh Sheaves for Projective Schemes

As for each A-module we can associate a quasi-coherent sheaf on $\operatorname{Spec} A$, we may extend this notion to graded modules on graded rings and their projective spectra.

Remark 2.2.1. For simplicity, we assume in this section that a graded ring is $\mathbb{N}$-graded and generated by degree 1. A graded module will however always be $\mathbb{Z}$-graded.

Lemma 2.2.2. *Let $S = \bigoplus_{n\geq 0} S_n$ be a graded ring, $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a graded module over S. Then for $f \in S$, the homogeneous localization $M_{(f)} := (M_f)_0$ is an $S_{(f)}$-module. Furthermore, if $f, g \in S_1$. Then $M_{(f)}[(g/f)^{-1}]$, $M_{(g)}[(f/g)^{-1}]$ and $M_{(fg)}$ are canonically isomorphic as $S_{(fg)}$-modules under the identification $S_{(f)}[(g/f)^{-1}] \cong S_{(g)}[(f/g)^{-1}] \cong S_{(fg)}$.*

Definition 2.2.3. Let S be a graded ring, M a graded S-module. We define the **quasi-coherent sheaf $\widetilde{M}$ associated to M on** $\operatorname{Proj} S$ to be the sheaf $\widetilde{M_{(f)}}$ on each homogeneous principal open $D_+(f)$. By 2.2.2 this definition coincides on intersections of principal opens.

Lemma 2.2.4. *Let S be a graded ring, M be a graded S-module. Then there is a canonical isomorphism $(\widetilde{M})_{\mathfrak{p}} \cong M_{(\mathfrak{p})}$, where $(\widetilde{M})_{\mathfrak{p}}$ is the stalk of $\widetilde{M}$ at $\mathfrak{p} \in \operatorname{Proj} S$, and $M_{(\mathfrak{p})} := (M_p)_0$ is the homogeneous localization of M at $\mathfrak{p}$.*

Lemma 2.2.5. *Let S be a graded ring, M, N be graded S-modules, $f : M \to N$ a graded module homomorphism. Then f induces a morphism of $\mathcal{O}_{\operatorname{Proj} S}$-modules $\widetilde{M} \to \widetilde{N}$.*

Definition 2.2.6. Let S be a graded ring, n an integer. We define the **twisting sheaf** $\mathcal{O}(n)$ on $\operatorname{Proj} S$ to be the quasi coherent sheaf $\widetilde{S(n)}$, where $S(n)$ is the graded S-module with grading $S(n)_m = S_{n+m}$. If $\mathcal{M}$ is any sheaf of $\mathcal{O}_{\operatorname{Proj} S}$-modules, we write $\mathcal{M}(n)$ for $\mathcal{M} \otimes \mathcal{O}(n)$.

Lemma 2.2.7. *The twisting sheaf* $\mathcal{O}(n)$ *is locally free of rank* 1*.*

We will study more about locally free sheaves of rank 1 in the next section.

Lemma 2.2.8. *For a graded* S*-module* M*, we have* $\widetilde{M}(n) \cong \widetilde{M(n)}$*. In particular,* $\mathcal{O}(n) \otimes \mathcal{O}(m) \cong \mathcal{O}(n+m)$*.*

Lemma 2.2.9. *Let* S *be a graded ring,* M, N *be graded* S*-modules. Assume there exists an integer* n_0 *such that the graded submodules* $\bigoplus_{n \geq n_0} M_n, \bigoplus_{n \geq n_0} N_n$ *are isomorphic, then the associated sheaves* $\widetilde{M}, \widetilde{N}$ *on* $\operatorname{Proj} S$ *are isomorphic.*

Proposition 2.2.10. *Let* S *be a graded ring,* I *a homogeneous ideal of* S*. Then the closed subscheme corresponding to* $\widetilde{I}$ *in* $\operatorname{Proj} S$ *is canonically isomorphic to* $\operatorname{Proj} S/I$*.*

Remark 2.2.11. In particular, if S is a finitely generated, graded k-algebra, and generated by degree 1, then $\operatorname{Proj} S$ can be considered as a closed subscheme of $\mathbb{P}^n_k$.

Definition 2.2.12. Let S be a graded ring, $\mathcal{M}$ a quasi-coherent sheaf on $\operatorname{Proj} S$. We define $\Gamma_* \mathcal{M} := \bigoplus_{n \in \mathbb{Z}} \Gamma(\operatorname{Proj} S, \mathcal{M}(n))$ to be the **graded S-module associated to $\mathcal{M}$**.

Remark 2.2.13. It is indeed nontrivial that $\Gamma_* \mathcal{M}$ has a graded S-module structure.

Lemma 2.2.14. *Let* S *be a graded ring,* M *a graded* S*-module. Then there exists an* n_0 *sufficiently large such that* $\bigoplus_{n \geq n_0} \Gamma(\operatorname{Proj} S, \widetilde{M}(n)) \cong \bigoplus_{n \geq n_0} M_n$*.*

Lemma 2.2.15. *If* $S = k[X_0, \ldots, X_n]$*, then* $\Gamma_* \mathcal{O} = S$*.*

Lemma 2.2.16. *Let* S *be a graded ring,* $\mathcal{M}$ *a quasi-coherent sheaf on* $\operatorname{Proj} S$*. There is an isomorphism* $\mathcal{M} \cong \widetilde{\Gamma_* \mathcal{M}}$*.*

Proposition 2.2.17. *Let* S *be a graded ring. There is an equivalence of categories between the category of quasi-coherent sheaves on* $\operatorname{Proj} S$*, and the category of graded* S*-modules modulo the equivalence relation:* $M \sim N$ *if and only if* $\bigoplus_{n \geq n_0} M_n \cong \bigoplus_{n \geq n_0} N_n$ *for some* n_0*. The equivalence goes by sending* $\mathcal{M}$ *to* $\Gamma_* \mathcal{M}$ *and* M *to* $\widetilde{M}$*.*

Proposition 2.2.18. *We can find for each closed subscheme* Z *in* $\mathbb{P}^n_k$ *a homogeneous ideal* I *in* $k[X_0, \ldots, X_n]$ *such that* $\widetilde{I}$ *cuts out* Z*.*

2.3 Invertible Sheaves

The simplest class in all quasi-coherent sheaves might be the invertible sheaves. The theory of divisors (including theory of curves and surfaces and the intersection theory) has also strong relations with the invertible sheaves. So it is worth spending a whole section talking about invertible sheaves.

Definition 2.3.1. A quasi-coherent sheaf $\mathcal{L}$ on a scheme X is called an **invertible sheaf**, or a **line bundle**, if there exists a quasi-coherent sheaf, denoted $\mathcal{L}^{-1}$, such that $\mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_X$.

Proposition 2.3.2. *All invertible sheaves on a scheme X, modulo isomorphisms, has a structure of abelian groups, where the addition is tensor product. We call the group the* ***Picard group*** *of X, denoted* $\mathrm{Pic}X$

Lemma 2.3.3. *Let X, Y be schemes, $f : X \to Y$ a morphism, $\mathcal{L}$ an invertible sheaf on Y. Then $f^*\mathcal{L}$ is an invertible sheaf on X.*

Theorem 2.3.4. *Let $\mathcal{L}$ be a quasi-coherent sheaf on a scheme X. Then the following are equivalent:*
1. $\mathcal{L}$ is invertible.
2. $\mathcal{L}$ is locally free of rank 1.
3. $\mathcal{L} \otimes \mathfrak{Hom}(\mathcal{L}, \mathcal{O}_X) \cong \mathcal{O}_X$.

Theorem 2.3.5. *Let X be a scheme over a ring A. Then there is a one-to-one correspondence between morphisms from X to $\mathbb{P}^n_A$ and the tuples $(\mathcal{L}, s_0, \dots, s_n)$ modulo isomorphisms, where $\mathcal{L}$ is a line bundle, and $s_0, \dots, s_n$ are global sections that generate $\mathcal{L}$.*

Proof. [1] Theorem II.7.1. □

Remark 2.3.6. It is worth noting that the pullback of $\mathcal{O}(1)$ in the projective space along the morphism defined by $\mathcal{L}$ is just $\mathcal{L}$ itself, and the global sections x_i are sent to s_i.

Remark 2.3.7. To get a geometric image, consider the case $A = k$ is a field and x a closed point in X with residue field k. The s_i can be evaluated at x and has the value $s_i(x)$ in k. Since s_i generate $\mathcal{L}$ they cannot be identically 0 at one point. Then we just map x to $[s_0(x) : \cdots : s_n(x)]$.

Theorem 2.3.8. *Let k be a field. Then* $\mathrm{Pic}\mathbb{P}_k^n \cong \mathbb{Z}$*, by sending n to $\mathcal{O}(n)$.*

Proof. [2] Example 11.43. $\square$

Theorem 2.3.9. *Let X be a quasi-compact and quasi-separated scheme, $\mathcal{M}$ a quasi-coherent sheaf on X. Let $\mathcal{L}$ be a line bundle, $s \in \Gamma(X, \mathcal{L})$ a global section. Then*
1. Let $f \in \Gamma(X, \mathcal{M})$ be a global section with $f|_{D(s)} = 0$. Then there exists an integer n such that $f \otimes s^n = 0$, where $f \otimes s^n$ is considered as a global section of $\mathcal{M} \otimes \mathcal{L}^n$.
2. For each section $f \in \Gamma(D(s), \mathcal{M})$ there exists an integer n and a section $f_X \in \Gamma(X, \mathcal{M} \otimes \mathcal{L}^n)$ such that $f_X|_{D(s)} = f \otimes s^n$.

Proof. 1. Let $X = \bigcup_{i=1}^n U_i$ be an affine open cover. Then on each $U_i \cap D(s)$, s corresponds to an element s_i in $\mathcal{O}_X(U_i) := A_i$, and $U_i \cap D(s) \cong \mathrm{Spec}\, A_{i,s_i}$. Therefore $f|_{U_i \cap D(s)} = 0$ implies that $s_i^n \cdot f(U_i) = 0$. The n here is dependent of i but by choosing n large enough we can assume $s_i^n \cdot f(U_i) = 0$ for all i. Now $s_i^n s_i^n \cdot f(U_i)$ is precisely the restriction of $f \otimes s^n$ on U_i. The assertion then follows.
2. We succeeding the notations above. Clearly $D(s) = \bigcup_{i=1}^n U_i \cap D(s)$ is an affine open cover. As $U_i \cap D(s)$ is the same as $D(s_i)$ in U_i, $f|_{U_i \cap D(s)}$ has actually the form f_i/s_i^n for some $f_i \in \Gamma(U_i, \mathcal{M} \otimes \mathcal{L}^n) \cong \Gamma(U_i, \mathcal{M})$. Hence $f_i|_{U_i \cap D(s)} = f \otimes s^n|_{U_i \cap D(s)}$. Moreover, by unifying the n for all i (and also adjusting f_i by multiplying with some power of s_i), $(f_i|_{U_i \cap U_j} - f_j|_{U_i \cap U_j})|_{D(s) \cap U_i \cap U_j}$ is zero. Since $U_i \cap U_j$ is quasi-compact and quasi-separated, we may use 1. and conclude that there exists an integer m such that $(f_i|_{U_i \cap U_j} - f_j|_{U_i \cap U_j}) \otimes (s|_{U_i \cap U_j})^m = 0$. Hence $f_i \otimes s^m$ glue together to be a global section in $\Gamma(X, \mathcal{M} \otimes \mathcal{L}^{n+m})$, which gives $f \otimes s^{n+m}$ after the restriction. $\square$

Corollary 2.3.10. *Let A be a graded ring finitely generated in degree 1, $\mathcal{M}$ a quasi-coherent sheaf of finite type on* $\mathrm{Proj}\, A$*. Then there exists an integer n_0 such that for all $n \geq n_0$ the sheaf $\mathcal{M}(n)$ is finitely globally generated.*

We always hope that a proper scheme over A can be embedded as a closed immersion into some projective space: $\iota : X \hookrightarrow \mathbb{P}_A^n$. Recall that to give a morphism into $\mathbb{P}_A^n$ is the same as to give a line bundle $\mathcal{L}$ and $n+1$ global sections. In order to characterize the

line bundles which indeed give a closed immersion, we observe two properties of the line bundle $\iota^*\mathcal{O}(1)$ which do not hold in general if ι is not a closed immersion:
1. For each $s \in \Gamma(X, \iota^*\mathcal{O}(1))$, $D(s)$ is affine.
2. For a quasi-coherent, of finite type sheaf $\mathcal{M}$ on X, $\mathcal{M}(n)$ is globally generated for $n \gg 0$.(See 2.3.10)
These two properties motivates the following definitions of ample line bundles. We will see that the two properties indeed almost characterized the bundles we want.

Definition 2.3.11. Let X be a scheme over a ring A, $\mathcal{L}$ an invertible sheaf on X. We call $\mathcal{L}$ a **very ample line bundle** relative to A if there exists a closed immersion $\iota : X \hookrightarrow \mathbb{P}^n_A$ such that $\mathcal{L} \cong \iota^*\mathcal{O}(1)$.

Definition 2.3.12. Let X be a quasi-compact scheme. We call an invertible sheaf $\mathcal{L}$ on X an **ample line bundle** if for each $x \in X$ there exists a $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ for some $n > 0$ such that $x \in D(s)$ and $D(s)$ is affine.

Lemma 2.3.13. *Let $\mathcal{L}$ be an ample line bundle on a scheme X. The following are equivalent:*
1. $\mathcal{L}$ is ample.
2. $\mathcal{L}^{\otimes n}$ is ample for all $n > 0$.
3. $\mathcal{L}^{\otimes n}$ is ample for some $n > 0$.

Lemma 2.3.14. *Let X be an affine scheme. Then every invertible sheaf is ample. Indeed, if s is a global section of the line bundle $\mathcal{L}^{\otimes n}$, then $D(s)$ is affine.*

Lemma 2.3.15. *Let $f : X \to Y$ be a locally closed immersion of schemes, $\mathcal{L}$ an ample line bundle on Y. Then $f^*\mathcal{L}$ is ample on X.*

Proposition 2.3.16. *Let X be a quasi-compact scheme with an ample line bundle. Then X is separated.*

Theorem 2.3.17 (criterion of ampleness)**.** *Let X be a quasi-compact scheme, $\mathcal{L}$ a line bundle. Then the following are equivalent:*
1. $\mathcal{L}$ is ample.
2. The open subsets $D(f)$ that are affine form a basis of topology, when f goes over all elements in $\Gamma(X, \mathcal{L}^{\otimes n})$ for all $n \geq 0$.

3. The open subsets $D(f)$ form a basis of topology, when f goes over all elements in $\Gamma(X, \mathcal{L}^{\otimes n})$ for all $n \geq 0$.
4. For all quasi-coherent, of finite type sheaf $\mathcal{M}$ on X, there exists an integer n_0, such that for all $n \geq n_0$, the quasi-coherent sheaf $\mathcal{M} \otimes \mathcal{L}^n$ is globally generated.

The next theorem due to Serre needs the theory of sheaf cohomology, but we still state it here for an easier looking up.

Theorem 2.3.18 (Serre's cohomological criterion of ampleness)**.** *Let X be a proper scheme over a Noetherian ring A, $\mathcal{L}$ a line bundle. Then the following are equivalent:*
1. $\mathcal{L}$ is ample.
2. For all quasi-coherent sheaves $\mathcal{M}$ of finite type on X, there exists an integer n_0, such that for all $n \geq n_0$ and all $i > 0$, $H^i(X, \mathcal{M} \otimes \mathcal{L}^{\otimes n}) = 0$.

Theorem 2.3.19. *Let X be a proper scheme over a ring A, $\mathcal{L}$ a line bundle. Then the following are equivalent:*
1.$\mathcal{L}$ is ample.
2.$\mathcal{L}^n$ is very ample relative to A for some n.
3.There exists an integer n_0 such that for all $n \geq n_0$, $\mathcal{L}^n$ is very ample relative to A.

2.4 Relative Spec and Proj

In this section we give the definition of relative Spec and relative Proj, and prove some funtorial properties. Recall that $\operatorname{Spec} A$ represents the functor $X \mapsto \operatorname{Hom}(A, \Gamma(X, \mathcal{O}_X))$ (1.2.26).

Theorem 2.4.1 (relative Spec)**.** *Let S be a scheme, $\mathcal{A}$ a quasi-coherent $\mathcal{O}_S$-algebra. Then there exists a scheme, denoted $\underline{\operatorname{Spec}}_S(\mathcal{A})$, called **relative Spec of $\mathcal{A}$ on** S, such that there is a bijection, functorial in X, when given a scheme morphism $f : X \to S$.*

$$\operatorname{Hom}_{S-\mathrm{Sch}}(X, \underline{\operatorname{Spec}}_S(\mathcal{A})) \cong \operatorname{Hom}_{\mathcal{O}_S-\mathrm{Alg}}(\mathcal{A}, f_*\mathcal{O}_X)$$

Remark 2.4.2. By Yoneda's lemma, the relative Spec of a certain $\mathcal{O}_S$-algebra is unique up to unique isomorphism.

To construct the right scheme we need one lemma.

Lemma 2.4.3. *Let X be a scheme, $\mathcal{A}$ a quasi-coherent $\mathcal{O}_X$-algebra, $U' \subseteq U \subseteq X$ two affine opens. We have a canonical map* $\operatorname{Spec}\mathcal{A}(U') \to \operatorname{Spec}\mathcal{A}(U)$ *induced by the restriction map, and canonical maps* $\operatorname{Spec}\mathcal{A}(U') \to U', \operatorname{Spec}\mathcal{A}(U) \to U$ *since $\mathcal{A}$ is an $\mathcal{O}_X$-algebra. Then the following diagram is Cartesian:*

$$\begin{tikzcd}
\operatorname{Spec}\mathcal{A}(U') \arrow[r] \arrow[d] & \operatorname{Spec}\mathcal{A}(U) \arrow[d] \\
U' \arrow[r] & U
\end{tikzcd}$$

Proof of Theorem 2.4.1. We construct $\underline{\operatorname{Spec}}_S(\mathcal{A})$ as follows. For each affine open $U \subseteq S$ the preimage $U \times_S \underline{\operatorname{Spec}}_S(\mathcal{A})$ is just $\operatorname{Spec}\mathcal{A}(U)$. By the previous lemma this construction glues well on overlaps.
Now let $f : X \to S$ be a morphism of schemes. Given a morphism of $\mathcal{O}_S$-algebras $\mathcal{A} \to f_*\mathcal{O}_X$, we get in particular homomorphism of rings $f_*\mathcal{O}_X(U) = \mathcal{O}_X(f^{-1}(U)) \to \mathcal{A}(U)$, which induces morphism of schemes $f^{-1}(U) \to \operatorname{Spec}\mathcal{A}(U)$. Let U goes over all affine opens of S, and glue these morphisms together using the property of sheaf morphisms we get a morphism $X \to \underline{\operatorname{Spec}}_S(\mathcal{A})$. We omit the verification that the following diagram commutes.
Conversely given a morphism of schemes $g : X \to \underline{\operatorname{Spec}}_S(\mathcal{A})$ such that the following diagram commutes,

$$\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dr, "f"'] & & \underline{\operatorname{Spec}}_S(\mathcal{A}) \arrow[dl, "\pi"] \\
& S &
\end{tikzcd}$$

we note that this gives a morphism of (not necessarily quasi-coherent) $\mathcal{O}_{\underline{\operatorname{Spec}}_S(\mathcal{A})}$-algebras $\mathcal{O}_{\underline{\operatorname{Spec}}_S(\mathcal{A})} \to g_*\mathcal{O}_X$, which gives after pushforward a morphism of (not necessarily quasi-coherent) $\mathcal{O}_S$-algebras $\pi_*\mathcal{O}_{\underline{\operatorname{Spec}}_S(\mathcal{A})} \to \pi_*g_*\mathcal{O}_X$. Now $\pi_*\mathcal{O}_{\underline{\operatorname{Spec}}_S(\mathcal{A})} \cong \mathcal{A}$, $\pi_*g_*\mathcal{O}_X \cong f_*\mathcal{O}_X$. The result follows.
We omit the verification that these two constructions are mutually inverse. $\square$

Remark 2.4.4. By the construction, the structure morphism $\pi : \underline{\operatorname{Spec}}_S(\mathcal{A}) \to S$ is affine. Explicitly, for each open affine $U \subseteq S$, the preimage $\pi^{-1}(U)$ is $\operatorname{Spec}\mathcal{A}(U)$. In particular, $\pi_*\mathcal{O}_{\underline{\operatorname{Spec}}_S(\mathcal{A})} = \mathcal{A}$.

Lemma 2.4.5. *Let S be a scheme, $\mathcal{A}, \mathcal{B}$ two quasi-coherent $\mathcal{O}_S$-algebras, and $\varphi : \mathcal{A} \to \mathcal{B}$ a morphism of $\mathcal{O}_S$-algebras. Then φ induces a morphism of schemes $\underline{\mathrm{Spec}}_S(\mathcal{B}) \to \underline{\mathrm{Spec}}_S(\mathcal{A})$ over S. Thus the relative Spec extends to a functor from the category of $\mathcal{O}_S$-algebras to the category of schemes over S.*

Proposition 2.4.6. *The relative Spec behaves well with pullbacks. Let X, Y be schemes, $f : X \to Y$ a morphism. $\mathcal{A}$ a quasi-coherent $\mathcal{O}_Y$-algebra. Then the following diagram, induced by the canonical morphism $\mathcal{A} \to f_* f^* \mathcal{A}$, is Cartesian.*

$$\begin{tikzcd}
\underline{\mathrm{Spec}}_{\mathcal{O}_X}(f^*\mathcal{A}) \arrow[r] \arrow[d] & \underline{\mathrm{Spec}}_{\mathcal{O}_Y}(\mathcal{A}) \arrow[d] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

We also construct the relative Proj via glueing. For simplicity all graded quasi-coherent $\mathcal{O}_S$-algebra is generated in degree 1.

Lemma 2.4.7. *Let S be a scheme, $\mathcal{A}$ a graded quasi-coherent $\mathcal{O}_S$-algebra. Let $U' \subseteq U \subseteq S$ be two affine opens. Then the homomorphism of graded rings $\mathcal{A}(U) \to \mathcal{A}(U')$ induces a morphism of schemes* $\mathrm{Proj}\,(\mathcal{A}(U')) \to \mathrm{Proj}\,(\mathcal{A}(U))$, *and the following diagram is Cartesian.*

$$\begin{tikzcd}
\mathrm{Proj}\,(\mathcal{A}(U')) \arrow[r] \arrow[d] & \mathrm{Proj}\,(\mathcal{A}(U)) \arrow[d] \\
U' \arrow[r] & U
\end{tikzcd}$$

Theorem 2.4.8. *Let S be a scheme, $\mathcal{A}$ a graded quasi-coherent $\mathcal{O}_S$-algebra. Then there exists a scheme, denoted $\underline{\mathrm{Proj}}_{\mathcal{O}_S}(\mathcal{A})$, called the **relative Proj of $\mathcal{A}$ on S**, together with a morphism $\pi : \underline{\mathrm{Proj}}_{\mathcal{O}_S}(\mathcal{A}) \to S$ such that for each affine open $U \subseteq S$, $\pi^{-1}(U) \cong$* $\mathrm{Proj}\,\mathcal{A}(U)$.

Proof. Just simply glue $\mathrm{Proj}\,\mathcal{A}(U)$ together. By the previous lemma they glue well. ☐

Remark 2.4.9. The relative Proj is actually also functorial like the relative Spec in some sense, but the precise formulation is too complicated. For more details see also [3] Tag 07ZF.

Lemma 2.4.10. *Let S be a scheme, $\mathcal{A}$ a graded quasi-coherent $\mathcal{O}_S$-algebra. For each affine open $U \subseteq S$, we have the tautological bundle $\mathcal{O}_{\pi^{-1}(U)}(1)$ on $\operatorname{Proj} \mathcal{A}(U)$. These bundles on each U agree on overlaps and therefore glue together. We denote the glued bundle with $\mathcal{O}_{\underline{\operatorname{Proj}}_{\mathcal{O}_S}(\mathcal{A})}(1)$.*

Proposition 2.4.11. *The relative Proj behaves well with pullbacks. Let X, Y be schemes, $f : X \to Y$ a morphism. $\mathcal{A}$ a graded quasi-coherent $\mathcal{O}_Y$-algebra. Then we have a morphism $\underline{\operatorname{Proj}}_{\mathcal{O}_X}(f^*\mathcal{A}) \to \underline{\operatorname{Proj}}_{\mathcal{O}_Y}(\mathcal{A})$, and the following diagram commutes and is Cartesian.*

$$\begin{tikzcd}
\underline{\operatorname{Proj}}_{\mathcal{O}_X}(f^*\mathcal{A}) \arrow[r] \arrow[d] & \underline{\operatorname{Proj}}_{\mathcal{O}_Y}(\mathcal{A}) \arrow[d] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Proposition 2.4.12. *Let S be a scheme, $\mathcal{A}$ a graded quasi-coherent $\mathcal{O}_S$-algebra, and $\mathcal{L}$ a line bundle on S. Define the* ***twisted algebra*** *$\mathcal{A} * \mathcal{L} := \bigoplus_{d \geq 0} \mathcal{A}_d \otimes \mathcal{L}^d$ with the obvious addition and multiplication. We have then an isomorphism of schemes $\phi : X * \mathcal{L} := \underline{\operatorname{Proj}}_{\mathcal{O}_S}(\mathcal{A} * \mathcal{L}) \cong X := \underline{\operatorname{Proj}}_{\mathcal{O}_S}(\mathcal{A})$ such that the following diagram commutes:*

$$\begin{tikzcd}
X * \mathcal{L} \arrow[rr, "\phi"] \arrow[dr, "\pi'"'] & & X \arrow[dl, "\pi"] \\
& S &
\end{tikzcd}$$

Moreover, we have $\mathcal{O}_{X*\mathcal{L}}(d) \cong \phi^*\mathcal{O}_X(d) \otimes \pi'^*\mathcal{L}^n$.

Sketch of proof. It follows from the fact that locally on $D_+(g) \subset X$ a section f/g^n of the structure sheaf can be mapped to $f \otimes s^n / g^n \otimes s^n$ and vice versa. For twisting sheaf $\mathcal{O}_{X*\mathcal{L}}(1)$ a section f/g^n should however be mapped to $f \otimes s^{n+1}/g^n \otimes s^n$. Thus we have to twist $\pi'^*\mathcal{L}^n$ for the right transition function. $\square$

Definition 2.4.13. Let S be a scheme. We define **the relative projective n-space** over S to be $\mathbb{P}^n_S := \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} S \cong \underline{\operatorname{Proj}}_{\mathcal{O}_S}(\operatorname{Sym} \mathcal{O}_S^{n+1})$.

Moreover we are interested in schemes over S that look locally like a relative projective n-space over S, which motivates us to the following definition.

Definition 2.4.14. Let S be a scheme, $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_S$-module. We call $\mathbb{P}(\mathcal{E}) := \underline{\mathrm{Proj}}_{\mathcal{O}_S}(\mathrm{Sym}\mathcal{E})$ the **projective bundle** of $\mathcal{E}$ over S. If $\mathcal{E}$ is locally free of rank $n+1$, we also call $\mathbb{P}(\mathcal{E})$ a $\mathbb{P}^n$**-bundle**.

Lemma 2.4.15. *Let S be a scheme, $\mathcal{E}$ a vector bundle on S of rank $n+1$. Then the $\mathbb{P}^n$-bundle $\mathbb{P}(\mathcal{E})$ deserves its name, i.e. for each open $U \subseteq X$ where $\mathcal{E}$ is trivial, $\pi^{-1}(U) \cong \mathbb{P}^n_U$, where $\pi : \mathbb{P}(\mathcal{E}) \to S$ is the structure morphism.*

Proposition 2.4.16. *Let S be a scheme, $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_S$-module, and $\mathcal{L}$ a line bundle on S. There is an isomorphism $\phi : \mathbb{P}(\mathcal{E} \otimes \mathcal{L}) \cong \mathbb{P}(\mathcal{E})$ such that the following diagram commutes:*

$$\begin{tikzcd}
\mathbb{P}(\mathcal{E} \otimes \mathcal{L}) \arrow[rr, "\phi"] \arrow[dr, "\pi'"'] & & \mathbb{P}(\mathcal{E}) \arrow[dl, "\pi"] \\
& S &
\end{tikzcd}$$

Moreover, we have $\mathcal{O}_{\mathbb{P}(\mathcal{E}\otimes\mathcal{L})}(1) \cong \phi^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \pi'^*\mathcal{L}$.

Theorem 2.4.17. *Let S be a scheme, $\mathcal{E}$ a vector bundle on S of rank $n+1$, X a scheme over S with structure morphism f. Then there is a one-to-one correspondence between morphisms $g : X \to \mathbb{P}(\mathcal{E})$ over S, and surjections $\eta : f^*\mathcal{E} \twoheadrightarrow \mathcal{L}$ modulo isomorphisms (see remark below), where $\mathcal{L}$ is a line bundle on X.*

$$\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dr, "f"'] & & \mathbb{P}(\mathcal{E}) \arrow[dl, "\pi"] \\
& S &
\end{tikzcd}$$

Remark 2.4.18. The isomorphism means a commutative diagram:

$$\begin{tikzcd}
& f^*\mathcal{E} \arrow[dl, two heads, "\eta"'] \arrow[dr, two heads, "\eta'"] & \\
\mathcal{L} \arrow[rr, "\sim"'] & & \mathcal{L}'
\end{tikzcd}$$

Sketch of proof. The bijection in one direction works by sending g to $f^*\mathcal{E} \twoheadrightarrow g^*\mathcal{O}(1)$, which is the pullback of the canonical surjection $\pi^*\mathcal{E} \twoheadrightarrow \mathcal{O}(1)$ along g.

Conversely assume we have $\eta : f^*\mathcal{E} \twoheadrightarrow \mathcal{L}$. Pick an affine open cover $S = \bigcup_i U_i$ such that $\mathcal{E}|_{U_i}$ are trivial. Then $\eta|_{f^{-1}(U_i)}$ degenerates to $\mathcal{O}^{n+1}_{f^{-1}(U_i)} \twoheadrightarrow \mathcal{L}|_{f^{-1}(U_i)}$, which gives $n+1$ global sections of $\mathcal{L}|_{f^{-1}(U_i)}$ and hence define a morphism $f^{-1}(U_i) \to \pi^{-1}(U_i) \cong \mathbb{P}^n_{U_i}$. By 2.3.5 the morphisms on each $f^{-1}(U_i)$ glue together and we obtain a morphism $X \to \mathbb{P}(\mathcal{E})$. ☐

Chapter 3

Flatness and Smoothness

In this chapter we extend the notion of smoothness in differential geometry to the algebraic schemes and study its properties and criteria.

3.1 Flatness and Faithfully Flat Descent

Recall that an A-module M is flat (resp. faithfully flat) if and only if the funtor $-\otimes_A M$ is exact (resp. faithful and exact). Again we want to extend the notion to all $\mathcal{O}_X$-modules on a scheme X. It will turn out that the flatness condition is crucial for a scheme to be relative smooth to another scheme.

Definition 3.1.1. Let X be a scheme. An $\mathcal{O}_X$-module $\mathcal{M}$ is called **flat** if the functor $\mathcal{O}_X\text{Mod} \to \mathcal{O}_X\text{Mod}, \mathcal{N} \mapsto \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{M}$ is exact.

Lemma 3.1.2. *Let X be a scheme, $\mathcal{M}$ an $\mathcal{O}_X$-module. Then $\mathcal{M}$ is flat if and only if the stalk $\mathcal{M}_x$ is a flat $\mathcal{O}_{X,x}$-module for all $x \in X$.*

Remark 3.1.3. Therefore an $\mathcal{O}_X$-module being flat is a stalk-local condition. In particular, if $\mathcal{M}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules, then $\mathcal{M}$ is flat if and only if there exists an affine cover $X = \bigcup_i U_i$ such that $\mathcal{M}(U_i)$ is flat over $\mathcal{O}_X(U_i)$, if and only if for each affine open $U \subseteq X$, $\mathcal{M}(U)$ is flat over $\mathcal{O}_X(U)$.

Definition 3.1.4. Let X, Y be schemes, $f : X \to Y$ a morphism. f is said to be **flat** if for all $x \in X$, $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,f(x)}$. f is said to be **faithfully flat** if f is flat and surjective

Remark 3.1.5. Note the definition of a flat morphism does not imply that $\mathcal{O}_X$ is flat as $\mathcal{O}_Y$-module.

Remark 3.1.6. The motivation of the definition of faithful flatness comes from the fact that a ring homomorphism $A \to B$ is faithfully flat if and only if B is flat over A and $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.

Lemma 3.1.7. *Let X, Y be schemes, $f : X \to Y$ a morphism. X is flat over Y if and only if there exist affine covers $X = \bigcup_i U_i, Y = \bigcup_i V_i$ such that $f(U_i) \subseteq V_i$ and $\mathcal{O}_X(U_i)$ is flat over $\mathcal{O}_Y(V_i)$, if and only if for each affine $U \subseteq X$ mapping into affine $V \subseteq X$, $\mathcal{O}_X(U)$ is flat over $\mathcal{O}_Y(V)$.*

Theorem 3.1.8. *Let X, Y be schemes of finite type over a field k, and Y irreducible $f : X \to Y$ a flat morphism. Then for any (even not closed) point $y \in Y$, the scheme theoretic fibre $f^{-1}(y)$ has dimension $\dim X - \dim Y$. Conversely if a fibre $f^{-1}(y)$ has dimension n, then $\dim X = \dim Y + n$.*

Proof. We only give the proof where X is irreducible. For a general proof see [1] Corollary III.9.6. First recall the algebraic fact that if $\phi : B \to A$ is a flat local homomorphism of local rings, then $\dim A = \dim B + \dim A/\mathfrak{m}_B A$. See [3] Tag 00ON for a proof.
We reduce to the case where $X = \operatorname{Spec} A, Y = \operatorname{Spec} B$ are affine. Then $\dim f^{-1}(y) = \dim A/\mathfrak{p}_y A$. Pick a maximal ideal $\mathfrak{m}$ in $A/\mathfrak{p}_y A$, it corresponds to a maximal ideal in A, which by abuse of notation we also denote with $\mathfrak{m}$. Then since A is catenary, $\dim A_\mathfrak{m}/\mathfrak{p}_y A_\mathfrak{m}$, $\dim A_\mathfrak{m} = \dim A$. Then we have $\dim A = \dim A_\mathfrak{m} = \dim B_{\mathfrak{p}_y} + \dim A_\mathfrak{m}/\mathfrak{p}_y A_\mathfrak{m} = \dim B + \dim A/\mathfrak{p}_y A$. As $\dim B = \dim Y$ and $\dim A = \dim X$ we are done.
The converse direction is clear. □

Remark 3.1.9. This shows that when we have a flat morphism $f : X \to Y$ and consider the fibre $f^{-1}(y)$, the fibre won't change illy when we slightly move the point y. A counter-exmaple is the projection of the cross $\operatorname{Spec} k[x, y]/(xy)$ onto the x-axis $\operatorname{Spec} k[x]$. The preimage of $x \neq 0$ is a closed point but the preimage of 0 is the y-axis.

We now start to talk about faithfully flat descent. The central question is, given a scheme X over a scheme S, a faithfully flat

morphism $f : S' \to S$ and a property P of $X \times_S S'$ (i.e. the scheme after base change), does X also have P?. The answer is yes for almost all scheme properties that we have encountered.
We can even generalize the notion and ask whether an $\mathcal{O}_X$-module $\mathcal{M}$ descends, i.e. is there an $\mathcal{O}_Y$-module $\mathcal{N}$ such that $\mathcal{M} \cong f^*\mathcal{N}$? Moreover we can even descend schemes through faithful flat morphisms. We will just list the descending properties and formulate the descent of modules. For a detailed discussion see [2] Chapter 14.

Theorem 3.1.10. *Let X, Y be schemes, $f : X \to Y$ a faithfully flat morphism. If X is either*
1. reduced,
2. normal,
3. nonsingular,
then so is Y.

Theorem 3.1.11. *Let X, Y be schemes over a scheme S and $f : X \to Y$ a morphism over S. Let S' be another scheme, $g : S' \to S$ be a base change morphism, and denote $f' : X' := X \times_S S' \to Y' := Y \times_S S'$ the schemes and morphisms after base change.*

$$\begin{tikzcd}
X' \arrow[rr, "f'"] \arrow[dr] \arrow[drr] & & Y' \arrow[dl] \arrow[drr] & & \\
& S' \arrow[drr, "g"'] & X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\
& & & S &
\end{tikzcd}$$

Assume g is surjective. If f' is either
1. surjective,
2. injective,
3. bijective,
then so is f.

Theorem 3.1.12. *Succeeding the notations above, and assume now g is quasi-compact and faithfully flat. If f' is either*
1. open,
2. closed,
3. a homeomorphism on the underlying topological spaces,
4. quasi-compact,
5. quasi-separated,

6. separated,
then so is f.

Theorem 3.1.13. *Succeeding the notations above. We continue to assume g is quasi-compact and faithfully flat. If f' is either*
1. locally of finite type,
2. of finite type,
3. locally of finite presentation,
4. of finite presentation,
5. an isomorphism,
6. a monomorphism,
7. a locally closed immersion,
8. an open immersion,
9. a closed immersion,
10. proper,
11. affine,
12. finite,
then so is f.

3.2 Formal Smoothness

We now introduce the notion of smoothness. We first give the most general definition of smoothness which seems to be less intuitive. Then we describe the properties that a smooth scheme possesses (mainly the properties of relative Kähler differentials in the next section), which shows that this definition really coincides with our intuition of smoothness in geometry.

Definition 3.2.1. A morphism of schemes $f : T \to T'$ is called a **first order thickening** or a **square zero extension** if f is a closed immersion and the corresponding sheaf of ideals $\mathcal{I}$ satisfies $\mathcal{I}^2 = 0$.

Remark 3.2.2. One should consider T' as the same scheme T but with a little more information of the normal vectors on T. A good example of a first order thickening that one should always keep in mind is $\operatorname{Spec} k \to \operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$, by sending ε to 0. We also call $\operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$ a point with normal vectors.

Lemma 3.2.3. *Let $f : T \to T'$ be a first order thickening of schemes. Then T and T' have the same underlying topological space.*

Definition 3.2.4. A morphism of schemes $f : X \to S$ is said to be **formally smooth** (resp. **formally unramified**, **formally étale**) if given any (not necessarily Cartesian) diagram:

$$\begin{array}{ccc} T & \longrightarrow & X \\ \downarrow & & \downarrow \\ T' & \longrightarrow & S \end{array}$$

there exists at least one (resp. at most one, exactly one) morphism $T' \to X$ making the diagram commute.

$$\begin{tikzcd} T \arrow[r] \arrow[d] & X \arrow[d] \\ T' \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

Remark 3.2.5. It is clear from definition that a morphism being formally smooth (resp. formally unramified, formally étale) satisfies COMP and BC. But one has to use sheaf cohomology to show that it is also LOCS and LOCT. For a proof see [3] Tag 0D0F.

Definition 3.2.6. A morphism of schemes $f : X \to S$ is said to be **smooth** (resp. **unramified**, **étale**), if it is formally smooth (resp. formally unramified, formally étale) and locally of finite presentation (resp. locally of finite type, locally of finite presentation).

Remark 3.2.7. In Grothendieck's EGA, unramified morphisms are asked to be locally of finite presentation. However we hope all closed immersions to be unramified (There are closed immersions whose sheaves of ideals are not of finite type), thus we weaken the condition for unramified morphisms.

Definition 3.2.8. Let X be a scheme over a field k, $x \in X$ a point with residue field k. We define the **Zariski cotangent space** at x to be $CT_xX := \mathfrak{m}_x/\mathfrak{m}_x^2$, where $\mathfrak{m}_x$ is the unique maximal ideal in the stalk $\mathcal{O}_{X,x}$. Note it has a k-vector space structure via the isomorphism $\mathcal{O}_{X,x}/\mathfrak{m}_x \cong k$. We define the **Zariski tangent space** to be the dual vector space $T_xX := (CT_xX)^\vee$.

Proposition 3.2.9. *Let X be a locally noetherian scheme over a field k, $x \in X$ a point with residue field k. Let $T = \operatorname{Spec} k \to$*

$T' = \operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$ *be the first order thickening. There is a one-to-one correspondence between the elements in* T_xX *and morphisms* $T' \to X$ *making the following diagram commute.*

$$\begin{tikzcd} T \arrow[r, "\bullet \mapsto x"] \arrow[d] & X \arrow[d] \\ T' \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} k \end{tikzcd}$$

Proof. To give a morphism $T' \to X$ is the same as to give a local ring homomorphism $\mathcal{O}_{X,x} \to k[\varepsilon]/(\varepsilon^2)$. Take a basis $\widetilde{f_1}, \cdots, \widetilde{f_n}$ in CT_xX, which by Nakayama's lemma lift to a generator set $f_1, \cdots, f_n$ in $\mathfrak{m}_x$. Given a local homomorphism $\mathcal{O}_{X,x} \to k[\varepsilon]/(\varepsilon^2)$ we know $f_i \mapsto a_i\varepsilon$ for some $a_i \in k$, which defines a linear map $CT_xX \to k$ by sending $\widetilde{f_i}$ to a_i. Conversely given a linear map $CT_xX \to k$, sending $\widetilde{f_i}$ to a_i we can define a local ring homomorphism $\mathcal{O}_{X,x} \to k[\varepsilon]/(\varepsilon^2), f_i \mapsto a_i\varepsilon$. □

3.3 Kähler Differentials

Proposition 3.2.9 gives us the intuition that smoothness has some relations with the tangent space. Hence We want to construct relative tangent space of a morphism $X \to S$ even when S is not affine and study its properties. This is the motivation of Kähler differentials. We start with an algebraic treatment of the derivation.

Definition 3.3.1. Let A, B be rings, $f : A \to B$ a ring homomorphism, M a B-module. An A-module homomorphism $d : B \to M$ is called an **A-linear derivation** if it satisfies:
1.(Annihilation) $d(f(a)) = 0$ for all $a \in A$.
2.(Leibniz's Rule) $d(ab) = a \cdot d(b) + b \cdot d(a)$ for all $a, b \in B$.

Remark 3.3.2. Note that d is not B-linear. Indeed, d is B-linear if and only if d is the zero homomorphism, since $d(b) = d(1 \cdot b) = b \cdot d(1) = 0$.

Remark 3.3.3. A good example one should keep in mind is the module of differentials of the polynomial ring $B := k[x, y]$ over k, which is defined as $M = B \cdot dx \bigoplus B \cdot dy$. The dx, dy are just formal symbols. The derivation is given by $d : B \to M, f \mapsto df = \frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} \cdot dy$

Proposition 3.3.4. *Let A, B be rings, $f : A \to B$ a ring homomorphism. There exists a B-module $\Omega^1_{B/A}$, called the **Kähler differential** of B relative to A, satisfying the following universal property:*
*1. There exists an A-linear derivation $d : B \to \Omega^1_{B/A}$, called the **universal derivation**.*
2. For a B-module M and an A-linear derivation d_M, there exists a unique B-module homomorphism $\phi : \Omega^1_{B/A} \to M$ such that $d_M = \phi \circ d$.
i.e. in the language of representable functors, there exists a natural bijection $\mathrm{Hom}_{B-\mathrm{Mod}}(\Omega^1_{B/A}, M) \cong \mathrm{Der}_A(B, M)$.

Proof. The construction of is completely formal. Define $\Omega^1_{B/A} := \bigoplus_{b \in B} B \cdot db / \sim$, where db are just formal symbols, and $\sim$ is the submodule generated by the relations $da, d(b_1+b_2) - db_1 - db_2, d(b_1b_2) - b_1db_2 - b_2db_1$ with $a \in A, b_1, b_2 \in B$. Define $d : B \to \Omega^1_{B/A}, b \mapsto db$. Given an A-linear derivation $d_M : B \to M$, we can define $\phi : \Omega^1_{B/A} \to M, db \mapsto d_M(b)$. We omit the verification that the construction is well-defined and universal. $\square$

Lemma 3.3.5. *Let A, B, A' be rings. Assume we have ring homomorphisms $A \to B$, $A \to A'$. Consider the push-out $B' := A' \otimes_A B$. There is a natural isomorphism of B'-modules $\Omega^1_{B'/A'} \cong B' \otimes_B \Omega^1_{B/A} \cong A' \otimes_A \Omega^1_{B/A}$.*

$$\begin{tikzcd}
\Omega^1_{B'/A'} \arrow[d, dash, dashed] & \Omega^1_{B/A} \arrow[d, dash, dashed] \\
B' & B \arrow[l] \\
A' \arrow[u] & A \arrow[l] \arrow[u]
\end{tikzcd}$$

Remark 3.3.6. In particular, $\Omega^1_{B[f^{-1}]/A[f^{-1}]} \cong \Omega^1_{B/A}[f^{-1}]$. And $\Omega^1_{[B/(f(I))]/[A/I]} \cong \Omega^1_{B/A}/I\Omega^1_{B/A}$.

Proposition 3.3.7. *Let A, B, C be rings, $f : A \to B, g : B \to C$ two ring homomorphisms. Then there exists an exact sequence of C-modules:*

$$C \otimes_B \Omega^1_{B/A} \to \Omega^1_{C/A} \to \Omega^1_{C/B} \to 0$$

Proof. [3] Tag 00RS. ☐

Proposition 3.3.8. *Succeeding the notations above, and assume that g is surjective with kernel $I \subseteq B$. Then $\Omega^1_{C/B} = 0$ and there exists an exact sequence of C-modules:*

$$I \otimes_B C \cong I/I^2 \to C \otimes_B \Omega^1_{B/A} \to \Omega^1_{C/A} \to 0$$

The left arrow is just the universal derivation of B relative to A tensoring with identity on C.

Proof. [3] Tag 00RU. ☐

Proposition 3.3.9. *Let A, B be rings, $f : A \to B$ a ring homomorphism. Write I for the kernel of the diagonal homomorphism $\Delta : B \otimes_A B \to B, b_1 \otimes b_2 \mapsto b_1 b_2$. Then I/I^2 has a B-module structure and identifies with the Kähler differential $\Omega^1_{B/A}$ via the A-linear derivation $b \mapsto b \otimes 1 - 1 \otimes b$.*

Proof. I/I^2 is indeed a B-module as $B \cong (B \otimes_A B)/I$. Indeed, I is generated by $b \otimes 1 - 1 \otimes b$ for $b \in B$. Given $b' \in B$ we can check $b' \cdot (b \otimes 1 - 1 \otimes b) = bb' \otimes 1 - 1 \otimes bb'$. We omit the verification that $d : b \mapsto b \otimes 1 - 1 \otimes b$ is indeed an A-linear derivation. Now given another A-linear derivation $d_M : B \to M$, we map $b \otimes 1 - 1 \otimes b$ to $d_M(b)$. ☐

With the identification above we can construct now the Kähler differential for general schemes.

Definition 3.3.10. Let X, S be schemes $f : X \to S$ a separated morphism. Let $\mathcal{I}$ be the ideal sheaf corresponding to the diagonal morphism $\Delta : X \to X \times_S X$. We define the **sheaf of differentials** of X relative to S, or the **cotangent sheaf** of X relative to S to be $\Omega^1_{X/S} := \Delta^* \mathcal{I}$.

Remark 3.3.11. Note that $\Omega^1_{X/Y}$ is quasi-coherent by definition.

Lemma 3.3.12. *Let X, S be schemes $f : X \to S$ a separated morphism. Take an affine open* $\operatorname{Spec} A \subseteq X$ *mapping to an affine open* $\operatorname{Spec} R \subseteq S$. *Then* $\Omega^1_{X/S}|_{\operatorname{Spec} A} \cong \Omega^1_{A/R}$.

Remark 3.3.13. One can also follow the definition of differentials of rings and define the differentials of schemes to be the glue of local differentials of rings and showing that it is precisely the pullback of the kernel of diagonal morphism.

Lemma 3.3.14. *Let X, S be schemes $f : X \to S$ a separated morphism. Let S' be another scheme and $g : S' \to S$ a morphism of schemes. Define $X' := X \times_S S'$ to be the scheme after base change. Denote the projection $X' \to X$ as g'. Then there is a natural isomorphism of $\mathcal{O}_{X'}$-modules $\Omega^1_{X'/S'} \cong g'^*\Omega^1_{X/S}$.*

$$\begin{tikzcd} \Omega^1_{X'/S'} \arrow[d, dashed, no head] & \Omega^1_{X/S} \arrow[d, dashed, no head] \\ X' & X \arrow[l, "g'"] \\ S' \arrow[u] & S \arrow[l] \arrow[u] \end{tikzcd}$$

Proposition 3.3.15. *4.0.11 Let X, Y, S be schemes, $f : X \to Y, g : Y \to S$ be morphisms. There exists an exact sequence of $\mathcal{O}_X$-modules:*

$$f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0$$

Proof. This is just the scheme version of 3.3.7 □

Proposition 3.3.16. *Succeeding the notations above, and assume that f is a closed immersion with corresponding sheaf of ideals $\mathcal{I}$ on X. Then $\Omega^1_{X/Y} = 0$ and there exists an exact sequence of $\mathcal{O}_X$-modules:*

$$f^*\mathcal{I} \to f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to 0$$

Proof. This is just the scheme version of 3.3.8 □

We now start to state the relations between smoothness and Kähler differential.

Lemma 3.3.17. *Let X be a separated scheme over a field k. Let $x \in X$ be a point with residue field k. There is an isomorphism of k-vector spaces $CT_xX \cong (\Omega^1_{X/k})_x \otimes_{\mathcal{O}_{X,x}} k$.*

Proposition 3.3.18. *Let X, S be schemes and $f : X \to S$ a morphism. Then f is formally unramified if and only if $\Omega^1_{X/S} = 0$.*

Proposition 3.3.19. *Let X, Y, S be schemes, $f : X \to Y, g : Y \to S$ be morphisms.*
1. If f is formally smooth, then the sequence of $\mathcal{O}_X$-modules is exact and splits locally:

$$0 \to f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0$$

2. If $g \circ f$ is formally smooth, and the sequence above is exact and splits locally, then f is formally smooth.

Corollary 3.3.20. *Let X, Y, S be schemes, $f : X \to Y, g : Y \to S$ be morphisms. If f is formally étale, then $f^*\Omega^1_{Y/S} \cong \Omega^1_{X/S}$.*

To give the proof of 3.3.19 we need a construction of first order thickening.

Definition 3.3.21. Let A be a ring, M an A-module. We define the ring $A[M]$ to be the A-module $A \oplus M$ with the obvious module addtion. Define the module multiplication as $(a_1 \oplus m_1) \cdot (a_2 \oplus m_2) := a_1a_2 \oplus (a_1m_2 + a_2m_1)$. Equivalently, $A[M] = \mathrm{Sym}^{\bullet}_A M/(\mathrm{Sym}^2_A M)$.

Lemma 3.3.22. *Let A be a ring, M an A-module. The projection $A[M] \to A$ is a first order thickening.*

Lemma 3.3.23. *Let A be a ring, M, N two A-modules, $f : M \to N$ an A-module homomorphism. Then f induces a ring homomorphism $A[M] \to A[N], a \oplus m \mapsto a \oplus f(m)$. Thus the construction $M \mapsto A[M]$ is a functor from $A-\mathrm{Mod}$ to the category of first order thickenings over A.*

Remark 3.3.24. Indeed, if we restrict the category of first order thickenings to the split first order thickenings (i.e. there exists a ring homomorphism $A \to A'$ such that $A \to A' \to A = \mathrm{Id}_A$), then the functor is an equivalence of categories.

Proposition 3.3.25. *Let R be a ring, A an R-algebra, M an A-module. There is a bijection between the R-linear derivations $A \to M$ and the ring homomorphisms $A \to A[M]$ that are R-linear and satisfy $A \to A[M] \to A = \mathrm{Id}_A$, where the last arrow is the usual projection.*

Proof. Given an R-linear derivation $d : A \to M$ we have an R-linear ring homomorphism $f : A \to A[M], a \mapsto a \oplus d(a)$. Conversely given an R-linear ring homomorphism $f : A \to A[M]$, it necessarily has the form $\mathrm{Id}_A \oplus d$ for some R-linear map $d : A \to M$. One can check d is indeed a derivation. We omit the verification that these two constructions are mutually inverse. $\square$

Proof of 3.3.19. We may reduce to the affine case $X = \operatorname{Spec} B, Y = \operatorname{Spec} A, S = \operatorname{Spec} R$. Denote $f^\sharp$ (resp. $g^\sharp$) to be the ring homomorphism $A \to B$ corresponding to f (resp. g). To prove 1. it suffices to construct a homomorphism $\delta : \Omega^1_{B/R} \to B \otimes_R \Omega^1_{A/R}$ such that $\delta \circ \rho = \mathrm{Id}_{B \otimes_R \Omega^1_{A/R}}$, where $\rho : b \otimes da \mapsto b \cdot df^\sharp(a)$ is the homomorphism $B \otimes_R \Omega^1_{A/R} \to \Omega^1_{B/R}$ corresponding to the second arrow in sequence. ρ rises to a ring homomorphism $B[B \otimes_R \Omega^1_{A/R}] \to B[\Omega^1_{B/R}]$. We denote the ring homomorphism by abuse of notation also with ρ. Note that ρ is a first order thickening, since the kernel of ρ lies completely in $\Omega^1_{B/R}$. Consider the diagram:

$$\begin{tikzcd}
B[\Omega^1_{B/R}] & & B \arrow[ll, "b \mapsto b \oplus db"'] \arrow[dll, dashed] \\
B[B \otimes_R \Omega^1_{A/R}] \arrow[u, "\rho"] & & A \arrow[ll, "a \mapsto f^\sharp(a) \oplus (1 \otimes da)"] \arrow[u, "f^\sharp"']
\end{tikzcd}$$

Since $f^\sharp$ is formally smooth we get a lift $B \to B[B \otimes_R \Omega^1_{A/R}]$. The lift is R-linear, and its compose with the projection $B[B \otimes_R \Omega^1_{A/R}] \to B$ is identity on B. Therefore we get an R-linear derivation $\lambda : B \to B \otimes_R \Omega^1_{A/R}$, and the dotted arrow is just $b \mapsto b \oplus \lambda(b)$. By the universal property of Kähler differentials we obtain a B-module homomorphism $\delta : \Omega^1_{B/R} \to B \otimes_R \Omega^1_{A/R}, db \mapsto \lambda(b)$. One sees now $\delta \circ \rho(b \otimes da) = \delta(b \cdot df^\sharp(a)) = b \otimes \lambda(df^\sharp(a)) = b \otimes da$.
For the 2. part we may by restriction to smaller affines assume that the sequence splits. Consider the following diagram:

$$\begin{tikzcd}
C/I & B \arrow[l] \arrow[dl, dashed, "u'"] \\
C \arrow[u] & A \arrow[l, "v"] \arrow[u, "f^\sharp"'] \\
 & R \arrow[ul] \arrow[u, "g^\sharp"']
\end{tikzcd}$$

where $I^2 = 0$. Since $f^\sharp \circ g^\sharp$ is formally smooth we get a lift u' such that $u' \circ f^\sharp \circ g^\sharp = v \circ g^\sharp$. But the whole diagram does not necessarily commute. One can check $v - u' \circ f^\sharp$ is an R-linear derivation from A to I. If we find a lift $u : B \to C$, then similarly $u - u'$ is an R-linear derivation from B to I. Then we have $v - u' \circ f^\sharp = (u - u') \circ f^\sharp$. This shows it suffices to find an R-linear derivation $\delta : B \to C$ such that $v - u' \circ f^\sharp = \delta \circ f^\sharp$ and conclude that $u := u' + \delta$. The question now degenerates to: Given an R-linear derivation $\delta_A : A \to I$, can we find an R-linear derivation $\delta_B : B \to I$ such that $\delta_A = \delta_B \circ f^\sharp$? We know that $\mathrm{Hom}_{B-\mathrm{Mod}}(\Omega^1_{B/R}, I) \to \mathrm{Hom}_{B-\mathrm{Mod}}(B \otimes_A \Omega^1_{A/R}, I) \cong \mathrm{Hom}_{A-\mathrm{Mod}}(\Omega^1_{A/R}, I)$ is surjective since the sequence splits. But $\mathrm{Hom}_{B-\mathrm{Mod}}(\Omega^1_{B/R}, I)$ corresponds to R-linear derivations $B \to I$, $\mathrm{Hom}_{A-\mathrm{Mod}}(\Omega^1_{A/R}, I)$ corresponds to R-linear derivations $A \to I$. And we are done. ☐

Proposition 3.3.26. *Let X, Y be schemes, $f : X \to Y$ a morphism. If f is formally smooth, then $\Omega^1_{X/Y}$ is locally projective.*

Proof. The problem is local, so we may assume $X = \operatorname{Spec} B, Y = \operatorname{Spec} A$. Denote with $f^\sharp : A \to B$ the corresponding ring homomorphism. Take a surjection of B-modules $M' \to M$ and consider the diagram:

$$\begin{array}{ccc} B[M] & \xleftarrow{\mathrm{Id}\oplus\delta_M} & B \\ \uparrow & & \uparrow \\ B[M'] & \xleftarrow[f\oplus 0]{} & A \end{array}$$

Since f is formally smooth we find a lift $\mathrm{Id} \oplus \delta_{M'}$ from B to $B[M']$. Therefore, given a homomorphism $\Omega^1_{B/A} \to M$ of B-modules, it corresponds to an A-linear derivation $\delta_M : B \to M$, which can be lifted to an A-linear derivation $\delta_{M'} : B \to M'$, which corresponds to a homomorphism $\Omega^1_{B/A} \to M'$. This implies $\mathrm{Hom}_{B-\mathrm{Mod}}(\Omega^1_{B/A}, M') \to \mathrm{Hom}_{B-\mathrm{Mod}}(\Omega^1_{B/A}, M)$ is surjective. ☐

Remark 3.3.27. If f is also locally of finite presentation, i.e. smooth, then $\Omega^1_{X/Y}$ is also of finite type. A quasi-coherent module is locally projective and of finite type if and only if it is locally free of finite rank, See [3] Tag 00NX. Therefore if f is smooth, then $\Omega^1_{X/Y}$ is locally free of finite rank.

Proposition 3.3.28. *Let X, Y, S be schemes, $f : X \to Y$ a closed immersion with sheaf of ideals $\mathcal{I}$, $g : Y \to S$ a morphism.*
1. If $g \circ f$ is formally smooth, then the sequence of $\mathcal{O}_X$-modules is exact and splits locally:

$$0 \to f^*\mathcal{I} \to f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to 0$$

2. If g is formally smooth, and the sequence above is exact and splits locally, then $g \circ f$ is formally smooth.

Proof. We reduce to the affine case $S = \operatorname{Spec} R, Y = \operatorname{Spec} A, X = \operatorname{Spec} A/I$. Denote $g^\sharp$ to be the ring homomorphism $R \to A$ corresponding to g. Denote $\rho : A/I \otimes_A I \to A/I \otimes_A \Omega^1_{A/R}, a \otimes b \mapsto a \otimes db$ to be the A/I-module homomorphism corresponding to the second arrow in the sequence.
For the first part we need a homomorphism $\delta : A/I \otimes_A \Omega^1_{A/R} \to A/I \otimes_A I$ such that $\delta \circ \rho = \mathrm{Id}_{A/I \otimes_A I}$. Consider the following diagram:

$$\begin{tikzcd}
A/I & A/I \arrow[l, "\mathrm{Id}"'] \arrow[dl, dashed, "u"'] & A \arrow[l, "p"'] \arrow[dll, dashed, "p'"] \\
A/I^2 \arrow[u] & R \arrow[l] \arrow[u]
\end{tikzcd}$$

We get a lift u as $g \circ f$ is formally smooth. And the usual projection $p' : A \to A/I^2$ is another lift. One checks then $p' - u \circ p$ is an R-linear derivation $A \to I/I^2$, which gives an A-module homomorphism $\delta' : \Omega^1_{A/R} \to I/I^2$ sending da to $p'(a) - u \circ p(a)$ and hence an A/I-module homomorphism $\delta : A/I \otimes_A \Omega^1_{A/R} \to I/I^2, da \mapsto p'(a) - u \circ p(a)$. We have then $\delta \circ \rho(a \otimes b) = \delta(a \otimes db) = a \otimes (p'(b) - u \circ p(b)) = a \otimes p'(b) = a \otimes b$ for $b \in I/I^2$.
For the second part we set the diagram:

$$\begin{tikzcd}
B/J & A/I \arrow[l] & A \arrow[l] \arrow[dll, dashed, "v"'] \\
B \arrow[u] & R \arrow[l] \arrow[u] \arrow[ur, "g^\sharp"']
\end{tikzcd}$$

As g is formally smooth we have a lift v. To give a homomorphism $A/I \to B$ we need to modify v a little bit to v' such that $v'|_I = 0$. If v' exists, then $\delta := v' - v$ is an R-linear derivation such that $\delta|_I = -v|_I$. Note that $v|_{I^2} = 0$ as $v(I^2) \subseteq J^2 = 0$ and hence $-v|_I$ gives an A/I-module homomorphism $I/I^2 \to J$. But the map $\mathrm{Hom}_{A/I-\mathrm{Mod}}(A/I \otimes_A \Omega^1_{A/R}, J) \to \mathrm{Hom}_{A/I-Mod}(I/I^2, J)$ is surjective as the sequence is exact and splits. Therefore we get a homomorphism of A/I-modules $A/I \otimes_A \Omega^1_{A/R} \to J$ and therefore a homomorphism of A-modules $\Omega^1_{A/R} \to J$. This gives an R-linear derivation $\delta : A \to J$ with $\delta|_I = -v|_I$. $v' := \delta + v$ is then a ring homomorphism $A \to B$ with $v'|_I = 0$ and hence factors through A/I. □

3.4 Jacobian Criterion

We now start to state perhaps the most important smoothness criterion. But before that we need a little bit preparation.

Proposition 3.4.1. *Let X, S be schemes, $g : X \to S$ a morphism. g is smooth if and only if for each point $x \in X$ there exists a neighbourhood $x \in U$ and sections $f_1, \ldots, f_n$ such that the map f from U to $\mathbb{A}^n_S$ induced by $f_1, \ldots, f_n$ is étale.*

$$\begin{tikzcd} U \arrow[r, "f:X_i \mapsto f_i"] \arrow[dr, "g|_U"'] & \mathbb{A}^n_S \arrow[d] \\ & S \end{tikzcd}$$

In this case, $\Omega^1_{X/S}|_U \cong \bigoplus_{i=1}^n \mathcal{O}_U \cdot df_i$.

Proof. The if part is clear as $\mathbb{A}^n_S \to S$ is smooth. For the only if part we assume g to be smooth. Without loss of generality we assume $U = \operatorname{Spec} B, S = \operatorname{Spec} R$ are affine. Write $A := R[X_1, \ldots, X_n]$ We may by restriction to smaller affines assume that $\Omega^1_{B/R}$ is a free module as g smooth implies $\Omega^1_{X/S}$ is finite locally free. Say we have a basis $(\omega_1, \ldots, \omega_m)$ with $\omega_i = \sum_{j=1}^k b_j \cdot df_j$. Then by localizing at nonzero b_j's (we shrink the affine open $\operatorname{Spec} B$ even smaller) and linear cancellation we can assume $(df_1 \ldots, df_n)$ is a basis (n might differ from k). Then take the morphism $f : \operatorname{Spec} B \to \mathbb{A}^n_R = \operatorname{Spec} A$ induced by $f_1, \ldots, f_n$ We obtain immediately the isomorphism $\Omega^1_{B/R} \cong B \otimes_R \Omega^1_{A/R}$. We have the exact sequence

$$B \otimes_A \Omega^1_{A/R} \to \Omega^1_{B/R} \to \Omega^1_{B/A} \to 0$$

Hence $\Omega^1_{B/A}$ is 0, hence f is formally unramified. Furthermore the sequence

$$0 \to B \otimes_A \Omega^1_{A/R} \to \Omega^1_{B/R} \to \Omega^1_{B/A} \to 0$$

is exact and splits. By 3.3.19 we have f is formally smooth. f being of finte type is clear. □

Theorem 3.4.2 (Jacobian Criterion)**.** *Let $i : Z \to \mathbb{A}^n_R$ be a closed immersion locally of finite presentation with codimension r. For any point $z \in Z$ let $f_1, \dots, f_r$ be the sections in $\mathcal{O}_{X,z}$ that cut out Z locally. Then Z is smooth over R in a neighbourhood of z if and only if the Jacobi matrix $\mathcal{J} := \left(\frac{\partial f_i}{\partial X_j}(z)\right)_{i,j}$ has rank r in residue field $k(z)$.*

$$\begin{tikzcd} Z \arrow[rr, "i"] \arrow[dr] & & \mathbb{A}^n_R \arrow[dl] \\ & \operatorname{Spec} R & \end{tikzcd}$$

Proof. We need first of all a lemma.

Lemma 3.4.3. *Let A be a local ring with residue field k. An A-linear map $M : A^r \to A^n$ is injective and splits if and only if M is injective as a k-linear map $k^r \to k^n$.*

Proof. [4] Lemma 6.3. □

Denote $Z = \operatorname{Spec} B, A = R[X_1, \dots, X_n]$. Let I be the ideal corresponding to i, $\mathfrak{p}$ the prime ideal corresponding to z. Then $I/I^2 \otimes k(z) = (I/I^2)_\mathfrak{p}/\mathfrak{p}(I/I^2)_\mathfrak{p}$ is an r-dimensional vector space over $k(z)$. Take $f_1, \dots, f_r \in I/I^2$ such that they form a basis in $I/I^2 \otimes k(z)$. Note that they generate $I_\mathfrak{p}$ locally around z. Consider the diagram:

$$\begin{tikzcd} & \bigoplus_{i=1}^r B_\mathfrak{p} \cdot e_i \arrow[r, "\left(\frac{\partial f_j}{\partial X_i}\right)_{i,j}"] \arrow[d, "e_i \mapsto f_i"'] & \bigoplus_{i=1}^n B_\mathfrak{p} \cdot dX_i \arrow[d, "\cong"] & & \\ 0 \arrow[r] & (I/I^2)_\mathfrak{p} \arrow[r, "f_i \mapsto df_i = \sum_{j=1}^n \frac{\partial f_i}{\partial X_j} dX_j"'] & \left(B \otimes \Omega^1_{A/R}\right)_\mathfrak{p} \arrow[r] & \Omega^1_{B/R} \arrow[r] & 0 \end{tikzcd}$$

The second row is exact and splits. By the previous lemma the diagram keeps exact after tensoring with $k(z)$. But then the left vertical arrow becomes bijective, which implies the composition from $\bigoplus_{i=1}^r k(z) \cdot e_i$ to $k(z) \otimes \Omega^1_{A/R}$ is injective, hence $\mathcal{J}^T = \left(\frac{\partial f_j}{\partial X_i}\right)_{i,j} \otimes k(z)$ is injective in $k(z)$, hence it has rank r.

Conversely assume $\mathcal{J}$ has rank r in $k(z)$. Then $\left(\frac{\partial f_j}{\partial X_i}\right)_{i,j}$ is injective and splits. We have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i=1}^{r} B_{\mathfrak{p}} \cdot e_i \arrow[r, "\left(\frac{\partial f_j}{\partial X_i}\right)_{i,j}"] \arrow[d, two heads, "e_i \mapsto f_i"'] & \bigoplus_{i=1}^{n} B_{\mathfrak{p}} \cdot dX_i \arrow[d, "\cong"] & & \\
& (I/I^2)_{\mathfrak{p}} \arrow[r, "f_i \mapsto df_i = \sum_{j=1}^{n} \frac{\partial f_i}{\partial X_j} dX_j"'] & \left(B \otimes \Omega^1_{A/R}\right)_{\mathfrak{p}} \arrow[r] & \Omega^1_{B/R} \arrow[r] & 0
\end{tikzcd}$$

The left vertical arrow is an isomorphism after tensoring with $k(z)$. By Nakayama's lemma the arrow itself is already an isomorphism of. $B_{\mathfrak{p}}$-modules. Hence the homomorphism $f_i \mapsto df_i$ is injective and locally splits. Therefore z has a small smooth neighbourhood. $\square$

Remark 3.4.4. One should compare this amazing result with the Jacobian criterion in classical differential geometry. It shows that the definition of formal smoothness indeed grasps the kernel of smoothness.

3.5 Regularity and Smoothness

Recall that a scheme X is called nonsingular or regular if each stalk $\mathcal{O}_{X,x}$ is a regular local ring. With the geometric words, the dimension of the Zariski tangent space is equal the dimension of the local ring. We will see in this section that regularity and flatness are closely related to smoothness. The proofs are however quite technical and will be omitted. A general reference is [4] Lecture 6.

Theorem 3.5.1. *Let X be a scheme locally of finite type over a field k. Let $x \in X$ be a closed point with residue field $k(x)$ which is separable over k. If the stalk $\mathcal{O}_{X,x}$ is regular, then X is smooth in a neighbourhood of x.*

Proof. [4] Theorem 6.11. $\square$

Corollary 3.5.2. *Let X be a nonsingular scheme over a perfect field k (e.g. k is algebraically closed or has characteristic 0). Then X is smooth.*

Theorem 3.5.3. *Let X be a scheme over a field k. Then the following are equivalent.*
1. X is smooth over k.
2. X is geometrically nonsingular, i.e. for an arbitrary algebraically closed field l over k, the scheme $X \times_k l$ is nonsingular.
3. There exists one algebraically closed field l over k such that $X \times_k l$ is nonsingular.

Proof. [4] Theorem 6.8. □

Theorem 3.5.4. *Let X, S be schemes, $f : X \to S$ be a morphism locally of finite presentation. Then the following are equivalent.*
1. f is smooth.
2. f is flat and has smooth fibres.
3. f is flat and has smooth geometric fibres, i.e. for an arbitrary algebraically closed field k and an arbitrary scheme morphism $\operatorname{Spec} k \to S$, the pullback $X \times_{\operatorname{Spec} k} S \to \operatorname{Spec} k$ is smooth.

Chapter 4

Cohomology

In this chapter we develop the very important cohomological tools for study of coherent sheaves. We will use Grothendieck's effaceable δ-functors as our basic cohomology theory, whose properties and related propositions will be mentioned shortly without a proof. A reference can be found in [3] Tag 010P. There is a more general way to construct derived functors on arbitrary abelian categories using the theory of triangulated categories originated from algebraic topology. For details see the Appendix.
We then start to give the basic facts of cohomology theory using effaceable functors. The motivation is that we want to measure how inexact a left exact functor on the right side is. That is, given a left exact functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$, we want a family of functors $\mathcal{H}^i : \mathcal{A} \to \mathcal{B}$ such that $\mathcal{H}^0 = \mathcal{F}$ and given any exact sequence $0 \to A \to B \to C \to 0$, there exists a long exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{H}^0(A) \arrow[r] & \mathcal{H}^0(B) \arrow[r] & \mathcal{H}^0(C) \arrow[dll, "\delta^0"] \\
 & \mathcal{H}^1(A) \arrow[r] & \mathcal{H}^1(B) \arrow[r] & \mathcal{H}^1(C) \arrow[dll, "\delta^1"] \\
 & \mathcal{H}^2(A) \arrow[r] & \cdots &
\end{tikzcd}$$

where δ^i are called connecting morphisms. Since $\mathcal{H}^0 = \mathcal{F}$, it extends the exact sequence $0 \to \mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{F}(C)$.

Definition 4.0.1. Let $\mathcal{A}, \mathcal{B}$ be abelian categories. A **(cohomological) δ-functor** is a collection of the following data:

1. A family of functors $\mathcal{F}^i : \mathcal{A} \to \mathcal{B}$, $i \geq 0$.
2. For each short exact sequence $0 \to A \to B \to C \to 0$ and each $i \geq 0$ a morphism $\delta^i_{A,B,C} : \mathcal{F}^i(C) \to \mathcal{F}^{i+1}(A)$.

They should satisfy:

1. For every shor exact sequence $0 \to A \to B \to C \to 0$ the following sequence is exact:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}^0(A) \arrow[r] & \mathcal{F}^0(B) \arrow[r] & \mathcal{F}^0(C) \arrow[dll, "\delta^0_{A,B,C}"] \\
& \mathcal{F}^1(A) \arrow[r] & \mathcal{F}^1(B) \arrow[r] & \mathcal{F}^1(C) \arrow[dll, "\delta^1_{A,B,C}"] \\
& \mathcal{F}^2(A) \arrow[r] & \cdots &
\end{tikzcd}$$

2. For every morphism of short exact sequences $(A \to B \to C) \to (A' \to B' \to C')$ and every $i \geq 0$ the diagram below commutes:

$$\begin{tikzcd}
\mathcal{F}^i(C) \arrow[r, "\delta^i_{A,B,C}"] \arrow[d] & \mathcal{F}^{i+1}(A) \arrow[d] \\
\mathcal{F}^i(C') \arrow[r, "\delta^i_{A',B',C'}"'] & \mathcal{F}^{i+1}(A')
\end{tikzcd}$$

Remark 4.0.2. Note that by definition $\mathcal{F}^0$ is left exact.

Definition 4.0.3. Let $\mathcal{A}, \mathcal{B}$ be abelian categories, $(\mathcal{F}^i)_i, (\mathcal{G}^i)_i$ two δ-functors, a **morphism of δ-functors** $\mathcal{F} \to \mathcal{G}$ is a collection of natural transformations $t^i : \mathcal{F}^i \to \mathcal{G}^i$ such that for each $i \geq 0$ and every short exact sequence $0 \to A \to B \to C \to 0$ the following diagram commute:

$$\begin{tikzcd}
\mathcal{F}^i(C) \arrow[r, "\delta^i_{\mathcal{F}}"] \arrow[d, "t^i"'] & \mathcal{F}^{i+1}(A) \arrow[d, "t^{i+1}"] \\
\mathcal{G}^i(C) \arrow[r, "\delta^i_{\mathcal{G}}"'] & \mathcal{G}^{i+1}(A)
\end{tikzcd}$$

Definition 4.0.4. Let $\mathcal{A}, \mathcal{B}$ be abelian categories, $/cF : \mathcal{A} \to \mathcal{B}$ a left exact functor. We consider the category of δ-functors extending $\mathcal{F}$: the objects are δ-functors $(\mathcal{F}^i)_i$ such that $\mathcal{F}^0 = \mathcal{F}$, and the morphisms are morphisms $(t^i)_i$ of δ-functors with $t^0 = \mathrm{Id}$. A **universal δ-functor extending $\mathcal{F}$** is an initial object in this category. If $(\mathcal{F}^i)_i$ is a universal δ-functor extending $\mathcal{F}$, we also call it the **right derived functor** of $\mathcal{F}$, denoted with $R^i\mathcal{F}$.

Remark 4.0.5. By the universal property of initial objects, the universal δ-functor extending $\mathcal{F}$, if exists, is unique up to unique isomorphism.

Definition 4.0.6. Let $\mathcal{A}, \mathcal{B}$ be abelian categories, $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ a left exact functor. Let $(\mathcal{F}^i)_i$ be a universal δ-functor extending $\mathcal{F}$. We call $\mathcal{F}^i(A)$ the ***i*-th cohomology of A (relative to $\mathcal{F}$)** for an object $A \in \mathcal{A}$.

Definition 4.0.7. Let $\mathcal{A}, \mathcal{B}$ be abelian categories, $(\mathcal{F}^i)_i$ a δ-functor. An object $A \in \mathcal{A}$ is called **$\mathcal{F}$-acyclic**, if $\mathcal{F}^i(A) = 0$ for all $i \geq 1$.

Definition 4.0.8. Let $\mathcal{A}, \mathcal{B}$ be abelian categories. A functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is called **effaceable** if for any object $A \in \mathcal{A}$, there exists a monomorphism $A \hookrightarrow A'$ such that $\mathcal{F}(A') = 0$.

Theorem 4.0.9. *Let $(\mathcal{F}^i)_i$ be a δ-functor such that $\mathcal{F}^i$ are effaceable for all $i \geq 1$, then $(\mathcal{F}^i)_i$ is a universal δ-functor extending $\mathcal{F}^0$.*

Theorem 4.0.10. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories, $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ a left exact functor. Assume $\mathcal{A}$ has enough injectives. Then the universal δ-functor extending $\mathcal{F}$ exists. The construction goes as follows: For an object $A \in \mathcal{A}$, take an injective resolution of A, i.e. an exact sequence $0 \to A \to I_0 \to I_1 \to \cdots$. Define $\mathcal{H}^i(A)$ to be the i-th cohomology of the sequence $0 \to \mathcal{F}(I_0) \to \mathcal{F}(I_1) \to \cdots$. This is an effaceable delta-functor extending $\mathcal{H}^0 = \mathcal{F}$, hence universal.*

Sketch of proof. The hardest part is to show that it is well-defined. Indeed any choice of injective resolutions are homotopy equivalent and hence have the same cohomology. It is effaceable because given an injective object I, there is an injective resolution $0 \to I \to I \to 0$. By construction $\mathcal{H}^i(I) = 0$ for $i \geq 1$. $\square$

Proposition 4.0.11. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories, $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ a functor having a right derived functor. If for an object $A \in \mathcal{A}$ there exists a resolution $0 \to A \to C_0 \to C_1 \to \cdots$ where C_i are $R^i\mathcal{F}$-acyclic, then the i-th cohomology of the sequence $0 \to \mathcal{F}(C_0) \to \mathcal{F}(C_1) \to \cdots$ coincides with $R^i\mathcal{F}(A)$.*

Remark 4.0.12. This proposition implies to compute $\mathcal{F}^i(A)$ it suffices to use an $\mathcal{F}$-acyclic resolution , chop off the term A and compute the cohomology of $\mathcal{F}^0(C^\bullet)$.

Remark 4.0.13. We have also the dual notions for right exact functors: coeffaceable, projective resolutions, left derived functors and so on. We omit the constructions here.

4.1 Sheaf Cohomology

We now consider the left exact functor $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$ from the category of $\mathcal{O}_X$-modules on a scheme X to the category of $\Gamma(X, \mathcal{O}_X)$-modules.

Lemma 4.1.1. *The derived functor of $\Gamma(X, -)$ coincides with the derived functor of $\Gamma(X, F(-))$ from the category of $\mathcal{O}_X$-modules to the category of abelian groups, where F is the forgetful functor from the category of $\mathcal{O}_X$-modules to the category of sheaves of abelian groups.*

Definition 4.1.2. Let X be a scheme, $\mathcal{F}$ an $\mathcal{O}_X$-module. We define $H^i(X, \mathcal{F}) := R^i\Gamma(X, \mathcal{F})$ to be the ***i*-th sheaf cohomology** of $\mathcal{F}$, where $R^i\Gamma$ is the i-th derived funtor of $\Gamma(X, -)$.

Remark 4.1.3. Do not mix the sheaf cohomology of a single object of $\mathcal{O}_X$-module with the cohomology of a complex with values in $\mathcal{O}_X$-modules. There are however indeed lots of relations between them. For details see the Appendix.

Proposition 4.1.4. *Let X be a scheme and $\mathcal{M}$ an injective $\mathcal{O}_X$-module. Then $\mathcal{M}$ is **flasque**, i.e. for any open $V \subseteq U$ in X the restriction map $\mathcal{M}(U) \to \mathcal{M}(V)$ is surjective.*

Proof. We have the natural identification $\mathcal{M}(U) \cong \mathrm{Hom}((i_U)_!\mathcal{O}_U, \mathcal{M})$, where $(i_U)_!$ is the extension by zero, i.e. for a sheaf $\mathcal{F}$ on U, $(i_U)_!\mathcal{F}(W) = \mathcal{F}(W)$ for $W \subseteq U$ and $(i_U)_!\mathcal{F}(W) = 0$ else. Then there is an injection $(i_V)_!\mathcal{O}_V \hookrightarrow (i_U)_!\mathcal{O}_U$. As $\mathcal{M}$ is injective, the induced map $\mathrm{Hom}((i_U)_!\mathcal{O}_U, \mathcal{M}) \to \mathrm{Hom}((i_V)_!\mathcal{O}_V, \mathcal{M})$ is surjective. By the identification above we get the result. $\square$

Theorem 4.1.5. *Let X be a scheme. The category of $\mathcal{O}_X$-modules has enough injectives.*

Proof. [1] Proposition III.2.2. □

Proposition 4.1.6. *Let $\mathcal{F}$ be a flasque sheaf on a scheme X, then $H^i(X, \mathcal{F}) = 0$ for $i \geq 1$, i.e. $\mathcal{F}$ is Γ-acyclic.*

Proof. [1] Proposition III.2.5. □

Remark 4.1.7. By 4.0.11, the cohomology of a flasque resolution of an $\mathcal{O}_X$-module also computes its sheaf cohomology.

Lemma 4.1.8. *Let X be an affine scheme. For any quasi-coherent sheaf $\mathcal{M}$ on X, its higher cohomology vanishes, i.e. $H^i(X, \mathcal{M}) = 0$ for all $i \geq 1$.*

Next we state the theory of Čech cohomology, which is a very important tool to compute the sheaf cohomology.

Definition 4.1.9. Let n be an integer and $I, J \subseteq \{0, \ldots, n\}$ two subsets, we write $I \leq J$ if $I \subseteq J$. Given a topological space X and $n+1$ opens $U_0, \ldots, U_n$, We define $U_I := \bigcap_{i \in I} U_i$. Given an abelian sheaf $\mathcal{F}$ (i.e. sheaf with values in an abelian category), and two indices $I \leq J$, we define $d_{I,J}$ to be the restriction map $\mathcal{F}(U_I) \to \mathcal{F}(U_J)$.

Definition 4.1.10. Let X be a scheme, $X = \bigcup_{i=0}^n U_i$ a finite open cover, denoted $\mathfrak{U}$, $\mathcal{F}$ an abelian sheaf on X. The k-th Čech group of $\mathcal{F}$ is defined as $\check{C}^k(\mathfrak{U}; \mathcal{F}) := \bigoplus_{|I|=k+1} \Gamma(U_I, \mathcal{F})$. We define the **Čech differential map** $d_k : \check{C}^k(\mathfrak{U}; \mathcal{F}) \to \check{C}^{k+1}(\mathfrak{U}; \mathcal{F})$ as

$$\Gamma(U_I, \mathcal{F}) \to \Gamma(U_J, \mathcal{F}) = \begin{cases} (-1)^j d_{I,J}, & \text{if } J = I \cup \{j\} \\ 0, & \text{else} \end{cases}$$

Remark 4.1.11. Note that to give a map from a finite direct sum to a finite direct sum is the same as to give maps between each pair of components, as what we did in the definition.

Lemma 4.1.12. $d_{k+1} \circ d_k = 0$. *It follows then the Čech groups with the differentials is a complex.*

Definition 4.1.13. The Čech groups together with the differentials are called the **Čech complex** relative to $\mathfrak{U}$ with coefficients in $\mathcal{F}$ and its k-th cohomology $\check{H}(\mathfrak{U}; \mathcal{F})$ is called the k-th **Čech cohomology** of $\mathcal{F}$ relative to $\mathfrak{U}$.

Remark 4.1.14. The motivation of Čech cohomology should be considered as: How is the sheaf enriched when we restrict the sheaf to smaller opens? The Čech cohomology gives a nice layer cut of collections of local sections, avoiding them to be restrictions of some global sections by modulo out the image of the former Čech group, and ensuring that their restrictions to smaller opens vanish by taking the kernel of the differential map.

We next show that in good conditions the Čech cohomology agrees with the sheaf cohomology. Therefore the Čech cohomology gives a relatively easy way to compute the sheaf cohomology.

Lemma 4.1.15. *Let X be a scheme, $\mathfrak{U} : X = \bigcup_{i=0}^{n} U_i$ a finite open covering and $\mathcal{F}$ an abelian sheaf on X. We have $\check{H}^0(\mathfrak{U}; \mathcal{F}) \cong \Gamma(X, \mathcal{F})$.*

4.2 Ext Functor

4.3 Higher Direct Images

4.4 Serre Duality

4.5 Hilbert Polynomial

Chapter 5

Curves

5.1 Adic Spectrum

5.2 Divisors and Riemann-Roch

Appendix A

Homological Algebra

A.1 Category of Complexes

A.2 Derived Category

A.3 Derived Functor

Appendix B

Serre's GAGA

Bibliography

[1] Algebraic Geometry, R. Hartshorne.

[2] Algebraic Geometry I, U. Görtz, T. Wedhorn.

[3] Stacks Project, Columbia University.

[4] Lecture Notes of Algebraic Geometry II, P. Scholze.

[5] A Course in Commutative Algebra, G. Kemper.