# On the Bounded Negativity Conjecture

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# Einleitung

In dieser These untersuchen wir eine alte folklore Vermutung in der Schnitttheorie auf Flächen, die unter dem Namen Beschränkte Negativität Vermutung bekannt ist. Die präzise Formulierung lautet wie folgt:

**Vermutung** (Beschränkte Negativität Vermutung)**.** *Für jede glatte projektive Fläche $X$ existiert eine Zahl $b(X)$, sodass $C^2 \geq b(X)$ für alle irreduziblen reduzierten Kurven $C$ auf $X$.*

Wir bemerken dass die Zahl $b$ von $X$ abhängt. Es existiert keine universelle Schranke für alle Flächen. Zum Beispiel können wir eine glatte Kurve $C$ auf $X$ nehmen und beliebig viele Punkte auf $C$ aufblasen. Damit erhalten wir eine strikte Transformierte von mit einem beliebigen negativen Selbstschnitt. Wenn die Vermutung für eine Fläche $X$ gilt, sagen wir dass $X$ beschränkte Negativität besitzt.
Bis jetzt ist kein Gegenbeispiel in Charakteristik 0 bekannt. In Charakteristik $p$ lässt sich ein Gegenbeispiel durch den Frobenius Morphismus konstruieren.

**Proposition 1.1** ([Har77, Exercise V.1.10])**.** *Sei $C$ eine glatte projektive Kurve über einem Körper $k$ mit Charakteristik $p$. Wir betrachten den Endomorphismus $\Gamma := \mathrm{Id} \times F$ auf dem Produkt $C \times C$, wobei $F$ der $p$-Potenz Frobenius Morphismus $C \to C$ ist. Sei $\Delta$ die diagonale Kurve auf $C \times C$. Dann gilt*

$$\Gamma^n(\Delta) \cdot \Gamma^n(\Delta) = p^n(2 - 2g(C)).$$

*Insbesondere hat die Fläche $C \times C$ keine beschränkte Negativität, wenn das Geschlecht von $C$ größer als 1 ist.*

Es gibt drei interessante Beobachtungen in dem Gegenbeispiel. Erstens, die Fläche $C \times C$ ist vom allgemeinen Typ. Zweitens, die Folge der Kurven, dessen Selbstschnitt zu $-\infty$ tendiert, wird von einem Endomorphismus der Fläche induziert, der rein inseparabel ist. Drittens, alle Kurven in der Folge sind isomorph als abstrakte Schemata. Diese drei Fakten motivieren uns, die folgenden Fragen zu stellen:

1) Können wir eine Fläche von Charakteristik 0 konstruieren, die einen surjektiven Endomorphismus, der kein Isomorphismus ist, besitzt, sodass die Beschränkte Negativität Vermutung nicht für diese Fläche gilt?

2) Können wir Gegenbeispiele (in Charakteristik $p$) finden, wobei die Kodaira Dimension kleiner als 2 ist?

3) Können wir Flächen konstruieren, auf denen es unendliche viele Kurven von einem fixierten Geschlecht und einem fixierten Selbstschnitt gibt?

Außerdem interessieren wir uns dafür, ob und wie die beschränkte Negativität unter Morphismen von Flächen bewahrt wird. Es stellen sich nun auf natürliche Weise weitere Fragen:

4) Sei $X \to Y$ ein endlicher Morphismus. Können wir die beschränkte Negativität von $X$ (bzw. von $Y$) aus der beschränkten Negativität von $Y$ (bzw. von $X$) folgern?

5) Sei $X \to Y$ ein birationaler Morphismus, oder präziser die Aufblasung von einem Punkt. Können wir die beschränkte Negativität von $X$ (bzw. von $Y$) aus der beschränkten Negativität von $Y$ (bzw. von $X$) folgern?

Wir zerlegen die Inhalte in fünf Kapitel. In Kapitel 3 geben wir einige einfache Resultate über die Beschränkte Negativität Vermutung. Zuerst geben wir einen Beweis von Proposition 1.1. Dann beweisen wir, dass eine Fläche beschränkte Negativität besitzt, wenn der anti-kanonische Divisor nef oder $\mathbb{Q}$-effektiv ist. Wir geben auch eine positive Antwort zu einer Richtung von Frage 4 und Frage 5: Sei $X \to Y$ ein endlicher Morphismus oder die Aufblasung eines Punktes. Wenn $X$ beschränkte Negativität hat, hat $Y$ auch beschränkte Negativität.

In Kapitel 4 untersuchen wir Frage 1. Wir folgen [Nak10] und zeigen dass eine Fläche beschränkte Negativität besitzt, wenn die Fläche einen surjektiven separablen Endomorphismus besitzt, der kein Isomorphismus ist. Weiterhin zeigen wir, dass eine negative Kurve im Verzweigungslokus des Endomorphismus liegen muss. Damit geben wir eine bessere untere Schranke für Selbstschnitte von Kurven auf Flächen mit einem solchen Endomorphismus. Als Korollar (Siehe auch [Fuj02]) klassifizieren wir alle Flächen mit nicht negativer Kodaira Dimension, die einen surjektiven separablen Endomorphismus besitzen, der kein Isomorphismus ist.

In Kapitel 5 folgen wir [CdB21] und konstruieren eine rationale Fläche in Charakteristik $p$ mit unbeschränkter Negativität. Diese Fläche ist eine Aufblasung von $\mathbb{P}^2$. Wir erklären auch den Zusammenhang zwischen dieser Fläche und der in Proposition 1.1 konstruierten Fläche. Das Resultat hat mehrere Korollare:

1) Es wird gezeigt dass eine Richtung von Frage 5 in Charakteristik $p$ nicht gilt: Sei $X \to Y$ ein birationaler Morphismus und es habe $Y$ beschränkte Negativität, dann muss $X$ keine beschränkte Negativität besitzen.

2) Da jede Fläche einen endlichen Morphismus zu $\mathbb{P}^2$ hat, sehen wir dass jede Fläche in Charakteristik $p$ eine Aufblasung besitzt, auf der die Beschränkte Negativität Vermutung scheitert, was eine Antwort auf Frage 2 gibt.

3) Die Fläche ist das Pullback der Aufblasung von $\mathbb{P}^2$ in einem Punkt unter einem endlichen Morphismus. Die letztere Fläche besitzt beschränkte Negativität und gibt deswegen ein Gegenbeispiel zu einer Richtung von Frage 4 in Charakteristik $p$: Sei $X \to Y$ ein endlicher Morphismus und es habe $Y$ beschränkte Negativität, dann muss $X$ keine beschränkte Negativität besitzen.

In Kapitel 6 untersuchen wir Frage 3. Wir folgen [BHK$^+$13] und zeigen, dass für alle Paare von ganzen Zahlen $g \geq 0, m \geq g/2 + 1$ eine Fläche existiert, auf der es unendlich viele Kurven von Geschlecht $g$ und Selbstschnitt $-m$ gibt. In Charakteristik 0 kann das Resultat erweitert werden für alle Paare $g \geq 1, m \geq 2$ oder $g = 0, m \geq 1$.
In Kapitel 7 geben wir einige mögliche Verallgemeinerungen der Beschränkten Negativität Vermutung. In der ersten Sektion wird die Schwache Beschränkte Negativität Vermutung diskutiert, die behaputet dass der Selbstschnitt von unten beschränkt ist, wenn wir das Geschlecht der Kurven beschränken. Die Vermutung ist in Charakteristik $p$ wegen Proposition 1.1 falsch. Wir folgen [Hao19] und beweisen die Vermutung in Charakteristik 0. Die zweite Sektion behandelt die Harbourne Konstanten, die einen möglichen Ansatz zu Frage 5 bietet. Wir geben einige Beispiele dafür, wie die Untersuchung von Harbourne Konstanten und Geradenkonfigurationen auf Flächen verbunden sind. Die dritte Sektion führt die Gewichtete Beschränkte Negativität Vermutung ein, die schwacher als die originale Vermutung ist und in Verbindung zu lokalen Seshadri Konstanten steht.

# 1 Introduction

In this thesis we study an old, folklore conjecture in the intersection theory on surfaces, which is often referred to as the *Bounded Negativity Conjecture.* The precise formulation is as follows.

**Conjecture** (Bounded Negativity Conjecture)**.** *For each smooth, projective surface $X$, there exists a number $b(X)$ such that $C^2 \geq -b(X)$ for all irreducible and reduced curves $C$ on $X$.*

Note that the number $b$ depends on $X$. A universal bound for all surfaces does not exist. For example, we can take a smooth curve $C$ on $X$ and blow up $X$ at an arbitrary number of distinct points that lie on $C$ to obtain a strict transform of arbitrarily negative self intersection. If the conjecture holds for a certain surface $X$, we say that $X$ has bounded negativity.
No counterexamples are known in characteristic 0. In characteristic $p$, a simple counterexample can be constructed using the Frobenius morphism.

**Proposition 1.1** ([Har77, Exercise V.1.10])**.** *Let $C$ be a smooth projective curve over a field $k$ of characteristic $p$. Consider the product $C \times C$ and the endomorphism $\Gamma := \mathrm{Id} \times F$, where $F$ is the $p$-power Frobenius morphism $C \to C$. Then, the self intersection of the image of the diagonal $\Delta$ under $\Gamma^n$ is given by*

$$\Gamma^n(\Delta) \cdot \Gamma^n(\Delta) = p^n(2 - 2g(C)).$$

*In particular, if the genus of $C$ is larger than 1, the surface $C \times C$ does not have bounded negativity.*

There are three interesting observations that we can make in the counterexample. First, the surface $C \times C$ is of general type. Second, the sequence of the curves whose self intersections tend to $-\infty$ is induced by a surjective endomorphism of the surface which is purely inseparable. Third, all the curves in the sequence are isomorphic as abstract schemes. These three facts motivate us to ask the following questions:

1) Can we construct a surface of characteristic 0 admitting a surjective endomorphism that is not an isomorphism, on which the Bounded Negativity Conjecture does not hold?

2) Can we find counterexamples (in characteristic $p$) of surfaces where the Kodaira dimension is less than 2?

3) Can we construct surfaces on which there are infinitely many curves of a fixed genus and a fixed self intersection?

Moreover, we are interested in whether and how the bounded negativity is preserved under morphisms between surfaces. So it is natural to ask

4) Let $X \to Y$ be a finite morphism. Can we derive the bounded negativity of $X$ (resp. of $Y$) from the bounded negativity of $Y$ (resp. of $X$)?

5) Let $X \to Y$ be a birational morphism, or more specifically, the blow up of a point. Can we derive the bounded negativity of $X$ (resp. of $Y$) from the bounded negativity of $Y$ (resp. of $X$)?

We separate the contents into five chapters:
In Chapter 3 we state some easy results about the Bounded Negativity Conjecture. We first give a proof of Proposition 1.1. Then, we prove that a surface has bounded negativity if the anti-canonical divisor is nef or $\mathbb{Q}$-effective. We give also an affirmative answer to one direction of Question 4 and Question 5: If $X \to Y$ is a finite morphism or the blow up of a point and $X$ has bounded negativity, then $Y$ also has bounded negativity.
In Chapter 4 we study Question 1. Following [Nak10], we show that if a surface admits a surjective separable endomorphism that is not an isomorphism, then it has bounded negativity. Furthermore, we show that a negative curve has to be in the ramification locus of the endomorphism, and give a finer lower bound of the self intersection of curves on surfaces with such an endomorphism. As a corollary (See also [Fuj02]), we classify all surfaces with non-negative Kodaira dimension admitting a surjective separable endomorphism that is not an isomorphism.
In Chapter 5 we follow [CdB21] and construct a rational surface in characteristic $p$ with unbounded negativity. The surface is constructed as a blow up of $\mathbb{P}^2$. We state also the relation of this surface with the surface constructed in Proposition 1.1. The result has several corollaries:

1) It shows that one direction of Question 5 does not hold in characteristic $p$: If $X \to Y$ is a birational morphism and $Y$ has bounded negativity, then $X$ may not have bounded negativity.

2) Since every surface admits a finite morphism to $\mathbb{P}^2$, we see that every surface in characteristic $p$ has a blow up on which the Bounded Negativity Conjecture fails, answering Question 2.

3) The surface is the pullback of $\mathbb{P}^2$ blown up at a single point under a finite morphism. The latter surface has bounded negativity, which gives a counterexample to one direction of Question 4 in characteristic $p$: If $X \to Y$ is a finite morphism and $Y$ has bounded negativity, then $X$ may not have bounded negativity.

In Chapter 6 we study Question 3. We follow [BHK$^+$13] and show that for any pair of integers $g \geq 0, m \geq g/2 + 1$, there exists a surface with infinitely many curves on it of genus $g$ and self intersection $-m$. In characteristic 0, the result extends to all pairs $g \geq 1, m \geq 2$ or $g = 0, m \geq 1$.

In Chapter 7 we give some possible generalizations of the Bounded Negativity Conjecture. The first section discusses the Weak Bounded Negativity Conjecture, which asserts that the self intersection is bounded from below if we bound the genus of the curves that we pick on the surface. The conjecture is false in characteristic $p$ by Proposition 1.1. We follow [Hao19] and give a proof of the conjecture in characteristic 0. The second section deals with the Harbourne constants, which gives a possible approach to Question 5. We give some examples on how the study of Harbourne constants is related to line configurations on surfaces. The third section introduces the Weighted Bounded Negativity Conjecture, which is weaker than the original conjecture, and is related to the local Seshadri constants.

**Notation:**

We work over an algebraically closed field $k$ unless explicitly mentioned otherwise. A *surface* or a single *curve* will always be smooth projective unless mentioned otherwise. A *curve on a surface* denotes an effective divisor on the surface, so it might be singular, reducible or non-reduced. By a *negative curve* we mean an irreducible and reduced curve with negative self intersection on a surface. By a $(-1)$-*curve* we mean a curve isomorphic to $\mathbb{P}^1$ and having self intersection $-1$. The letter $p$ denotes always a positive prime number when used as the characteristic of a field.

$\mathbb{P}^n$: The $n$-dimensional projective space.

$\mathrm{Pic}(X)$: The Picard group of the surface $X$.

$C \cdot D$: The intersection number of the divisors $C$ and $D$.

$C^2$: The self intersection number of $C$.

$C \sim D$: Linear equivalence.

$C \sim_{\mathbb{Q}} D$: Linear equivalence of $\mathbb{Q}$-divisors.

$C \equiv D$: Numerical equivalence.

$c_k(E)$: The $k$-th Chern class of $E$.

$K_X$: The canonical divisor of $X$.

$\deg f$: The degree of the finite morphism $f$.

$N^1(X)$: $(\mathrm{Pic}(X)/\equiv) \otimes_{\mathbb{Z}} \mathbb{R}$, i.e. the group of $\mathbb{R}$-divisors on $X$ up to numerical equivalence.

$H^k(X, D)$: The $k$-th cohomology of $\mathcal{O}_X(D)$ on $X$

$h^k(X, D)$: The dimension of $H^k(X, D)$ as vector space

$\kappa(X)$: The Kodaira dimension of $X$.

$\rho(X)$: The Picard number of $X$.

$\mathrm{supp}\,(D)$: The support of the divisor $D$.

$\chi(X, D)$: The Euler characteristic of $\mathcal{O}_X(D)$ on $X$.

$g(C)$: The geometric genus of $C$.

$p_a(C)$: The arithmetic genus of $C$.

$\mathrm{Alb}(X)$: The Albanese variety of $X$.

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## Contents

# 2 Preliminaries

In this chapter we give some results on algebraic surfaces that will be frequently used in the next chapters.

## 2.1 Intersection theory of the blow up of a surface

The relations of the intersection theory of the blow up and the intersection theory of the surface itself plays an important role in this thesis. We state here the relation between the Picard groups and the intersection form on the blow up and the ones of the original surface. We refer to Section V.3 in [Har77] for details.

**Theorem 2.1** (Picard group of blow up)**.** *Let $X$ be a surface and $p \in X$ be a point. Let $\widetilde{X}$ be the blow up of $X$ at $p$ and $E$ be the exceptional divisor. Then the pullback $f^* : \mathrm{Pic}(X) \to \mathrm{Pic}(\widetilde{X})$ and the homomorphism $g : \mathbb{Z} \to \mathrm{Pic}(\widetilde{X}), 1 \mapsto E$ gives an isomorphism $\mathrm{Pic}(\widetilde{X}) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}$.*

*Proof.* [Har77, Proposition V.3.2] □

Under this identification, we can write every divisor on $\widetilde{X}$ as $f^*D + nE$, where $D$ is a divisor on $X$ and $n$ is an integer, up to linear equivalence. In this explicit situation the intersection theory of general varieties reduces to a simple form.

**Theorem 2.2** (Intersection pairing of blow up)**.** *Let $X$ be a surface and $p \in X$ be a point. Let $\widetilde{X}$ be the blow up of $X$ at $p$ and $E$ be the exceptional divisor. The intersection pairing on $\mathrm{Pic}(\widetilde{X})$ is given by*

$$\mathrm{Pic}(\widetilde{X}) \times \mathrm{Pic}(\widetilde{X}) \to \mathbb{Z}$$
$$(f^*D_1 + n_1E, f^*D_2 + n_2E) \mapsto D_1 \cdot D_2 - n_1n_2,$$

*where $D_1 \cdot D_2$ is the intersection on $X$. In particular, we have*

*1) $f^*D_1 \cdot f^*D_2 = D_1 \cdot D_2$,*

*2) $f^*D_1 \cdot E = 0$,*

*3) $E^2 = -1$*

*Proof.* [Har77, Proposition V.3.2] □

## 2.2 Zariski decomposition

The Zariski decomposition gives a possible way to separate a divisor into a nef part and a negative part.

**Theorem 2.3** (Zariski decomposition). *Let $D$ be a $\mathbb{Q}$-effective divisor on a surface $X$, i.e. $D \sim_{\mathbb{Q}} \sum_i \alpha_i D_i$ for some positive rational numbers $\alpha_i$ and effective divisors $D_i$. Then there exist unique $\mathbb{Q}$-effective divisors $P$ and $N$ such that $D \sim_{\mathbb{Q}} P + N$ and*

1) *$P$ is nef*

2) *$D - N$ is $\mathbb{Q}$-effective. If $N$ is nonzero, it has negative definite intersection matrix. More precisely, let $N = \sum_i N_i$, where $N_i$ are irreducible curves on $X$. The intersection matrix is defined to be $(N_i \cdot N_j)_{i,j}$. In particular, $N^2 < 0$, and $N_i^2 < 0$ for all $i$.*

3) *$P$ is orthogonal to $N$, i.e. $P \cdot N_i = 0$ for all $i$.*

*Proof.* [Băd01, Theorem 14.14] □

## 2.3 Miyaoka-Yau inequality

We need this inequality for the proof of the Weak Bounded Negativity Conjecture.

**Theorem 2.4** (Logarithmic Miyaoka-Yau inequality). *Let $X$ be a complex surface. Let $C$ be a smooth curve on $X$ such that the logarithmic canonical divisor $K_X + C$ is big or $\mathbb{Q}$-effective. Then we have the following inequality:*

$$c_1^2(\Omega_X^1(\log C)) \leq 3c_2(\Omega_X^1(\log C)).$$

*Equivalently, $(K_X + C)^2 \leq 3(c_2(X) - 2 + 2g(C))$.*

*Proof.* The big case is [BBC+12, Theorem A.2.8]. The $\mathbb{Q}$-effective case is [Miy84, Corollary 1.2]. □

## 2.4 Finite morphisms to projective space

For any base point free divisor $D$ on a smooth projective surface, we can find a finite morphism to $\mathbb{P}^2$ such that the pullback of $\mathcal{O}(1)$ is $D$. More generally, we have the following proposition.

**Proposition 2.5.** *Let $X$ be a smooth projective variety of dimension $n$. Let $D$ be a base point free divisor on $X$. Then there exists a finite surjective morphism $f : X \to \mathbb{P}^n$ such that $f^*\mathcal{O}_{\mathbb{P}^n}(1) \cong D$.*

*Proof.* The base point free divisor $D$ defines a morphism $\phi_D : C \to \mathbb{P}^N$ to a projective space such that $\mathcal{O}(D) \cong \phi_D^*\mathcal{O}(1)$. The projection from one point $\alpha : \mathbb{P}^k \dashrightarrow \mathbb{P}^{k-1}$ has the property that $\mathcal{O}_{\mathbb{P}^k}(1) = \alpha^*\mathcal{O}_{\mathbb{P}^{k-1}}(1)$. By successive projections $\mathbb{P}^k \dashrightarrow \mathbb{P}^{k-1}$ from a general point, we obtain a finite surjective morphism $f : X \to \mathbb{P}^n$ with $f^*\mathcal{O}_{\mathbb{P}^n}(1) \cong D$. □

# 3 General Results

In this chapter we give a proof of Proposition 1.1 and collect some numerical conditions under which the Bounded Negativity Conjecture holds.

## 3.1 Proof of Proposition 1.1

Recall Proposition 1.1:

**Proposition 1.1** ([Har77, Exercise V.1.10]). *Let $C$ be a smooth projective curve over a field $k$ of characteristic $p$. Consider the product $C \times C$ and the endomorphism $\Gamma := \mathrm{Id} \times F$, where $F$ is the $p$-power Frobenius morphism $C \to C$. Then, the self intersection of the image of the diagonal $\Delta$ under $\Gamma^n$ is given by*

$$\Gamma^n(\Delta) \cdot \Gamma^n(\Delta) = p^n(2 - 2g(C)).$$

*In particular, if the genus of $C$ is larger than 1, the surface $C \times C$ does not have bounded negativity.*

*Proof.* We first calculate

$$\Delta^2 = \deg(\mathcal{O}(\Delta)|_\Delta) = -\deg(\mathcal{O}(-\Delta)|_\Delta) = -\deg K_C = 2 - 2g(C).$$

Observe that $\Gamma^n_* \Delta = \Gamma^n(\Delta)$, as the composition $C \to \Delta \to C$, where the second arrow is induced by the projection $C \times C \to C$, is $\mathrm{Id}_C$ by the definition of diagonal morphisms. This implies $\deg(C \to \Delta) = 1$.
Now, by the projection formula,

$$\Gamma^n(\Delta) \cdot \Gamma^n(\Delta) = \Gamma^n_* \Delta \cdot \Gamma^n_* \Delta = \Delta \cdot (\Gamma^n)^* \Gamma^n_* \Delta = \deg \Gamma^n \cdot \Delta^2 = p^n(2 - 2g(C)).$$

□

## 3.2 Bounded negativity with nef or $\mathbb{Q}$-effective anti-canonical divisor

When the anti-canonical divisor is nef or $\mathbb{Q}$-effective, the bounded negativity follows from the adjunction formula.

**Proposition 3.1.** *Let $X$ be a surface with nef anti-canonical divisor, then for each irreducible and reduced curve $C$ on $X$ we have $C^2 \geq -2$.*

*Proof.* By the adjunction formula,

$$C^2 = -K_X \cdot C + 2p_a(C) - 2 \geq -2.$$

□

**Proposition 3.2.** *Let $X$ be a surface with $\mathbb{Q}$-effective anti-canonical divisor. Take the Zariski decomposition $-K_X \sim_{\mathbb{Q}} P + N = P + \sum_j a_j N_j$, where $N_j$ are irreducible components of $N$. Then for all irreducible and reduced curves $C$ on $X$, we have $C^2 \geq \left(\min_j N_j^2\right) - 2$.*

*Proof.* The curve $C$ is either one of the $N_j$, or has positive intersection with $-K_X$. By the adjunction formula,

$$C^2 = -K_X \cdot C + 2p_a - 2 \geq \left(\min_j N_j^2\right) - 2.$$

□

## 3.3 Passing through finite morphisms or blow ups

We give a partial answer to Question 4 and Question 5 in the Introduction about the behaviour of bounded negativity under finite morphisms or blow ups.

**Proposition 3.3.** *Let $X, Y$ be surfaces and $f : X \to Y$ a finite morphism. If there exists an integer $b(X)$ such that for all irreducible and reduced curves $C$ on $X$ we have $C^2 \geq -b(X)$, then for all irreducible and reduced curves $D$ on $Y$ we have $D^2 \geq -\deg f \cdot b(X)$.*

*Proof.* For a curve $D$ on X we have $f_* f^* D = \deg f \cdot D$. The pullback $f^* D$ can be written as sum of curves $\sum_i n_i C_i$, where $C_i$ are irreducible reduced curves on $X$, and $n_i$ are integers such that $\sum_i n_i \leq \deg f$. By the projection formula,

$$\begin{aligned} D^2 &= \frac{1}{\deg f} f_* f^* D \cdot D = \frac{1}{\deg f} f^* D^2 \\ &\geq \frac{1}{\deg f} \sum_i n_i^2 C_i^2 \geq \frac{1}{\deg f} (\deg f)^2 (-b(X)). \end{aligned}$$

□

**Proposition 3.4.** *Let $\widetilde{X}$ be the blow up of a surface $X$ at a point. Assume that there exists an integer $b(\widetilde{X})$ such that for all irreducible and reduced curves $C$ on $\widetilde{X}$, we have $C^2 \geq -b(\widetilde{X})$. Then for all irreducible and reduced curves $D$ on $X$, we have $D^2 \geq -b(\widetilde{X})$*

*Proof.* This follows directly from the fact that for the strict transform $\widetilde{D}$ of $D$, we have $D^2 \geq \widetilde{D}^2$. □

# 4 Surfaces with surjective separable endomorphisms

The counterexample of Proposition 1.1 is based on a non-trivial surjective endomorphism, i.e. a surjective endomorphism that is not an isomorphism. The endomorphism in this example induces a inseparable field extension on the function field of $C \times C$. It turns out that it is this inseparability of the endomorphism making the bounded negativity fail. In this chapter, we show that if a surface (of arbitrary characteristic) admits a non-trivial surjective separable endomorphism, then it has bounded negativity.

## 4.1 Bounded negativity with non-trivial surjective separable endomorphisms

As the first step, we show that any surjective endomorphism of a surface is finite.

**Lemma 4.1** ([Fuj02, Lemma 2.3])**.** *Let $X$ be a surface admitting a non-trivial surjective endomorphism $f : X \to X$. The pullback of divisors induces an automorphism of vector spaces $f^* : N^1(X) \to N^1(X)$. Moreover, $f$ is finite.*

*Proof.* We first show that the pullback $f^* : N^1(X) \to N^1(X)$ is injective. Assume that there exists a divisor $D$ on $X$ such that $f^*D \equiv 0$, i.e. $0 = f^*D \cdot C = D \cdot f_*C$ for all curves $C$ on $X$. As $f$ is surjective, divisors of the form $f_*C$ generate the space $N^1(X)$, hence $D \equiv 0$.
Since $N^1(X)$ is a finite dimensional vector space of dimension $\rho(X)$, $f^*$ is isomorphism. If $f$ is not finite, then there exists a curve $C$ on $X$ contracted to a point via $f$. Take an ample divisor $H$ on $X$. Then, we can find an $\mathbb{R}$-divisor $D$ such that $H = f^*D$. By the projection formula $C \cdot H = f_*C \cdot D = 0$, a contradiction. □

We can show that any surface admitting a non-trivial surjective separable endomorphism has only finitely many negative curves, but before that we need a lemma in set theory.

**Lemma 4.2.** *Let $S$ be a set, and $T \subseteq S$ be a finite subset. Let $f : S \to S$ be an injective map. Moreover, assume that there exists an integer $m$ such that $f^m(t) = t$ for all $t \in T$. If for any $s \in S$, there exists an integer $k$ such that $f^k(s) \in T$, then $S$ is also finite.*

*Proof.* Write $f^{-l}(t) := \{s \in S \mid f^l(s) = t\}$ for the preimage set of $t$ under $f^l$. We deduce that for all $t \in T$, the set $\bigcup_{l \geq 0}^{\infty} f^{-l}(t)$ is a finite set, since $f$ is injective and $f^m|_T$ is the identity on $T$.
Assume that $S$ is infinite. By the drawer principle, there exists at least

one element $t \in T$, such that there exists an infinite sequence of elements $s_i \in S$ and integers $k_i$ with $f^{k_i}(s_i) = t$. But this is a contradiction to that $\bigcup_{l\geq 0}^{\infty} f^{-l}(t)$ is finite. □

**Proposition 4.3** ([Nak10, Lemma 3.1])**.** *Let $X$ be a surface admitting a non-trivial surjective separable endomorphism $f : X \to X$. Then there are only finitely many negative curves on $X$. In particular, $X$ has bounded negativity.*

*Proof.* Denote the set of negative curves on $X$ by $\mathrm{Neg}(X)$. We need to show that $\mathrm{Neg}(X)$ is a finite set.
By the projection formula, we have $f_* f^* C = \deg f \cdot C$ for all curves, hence $f_*$ is also an isomorphism on $N^1(X)$. We claim that $f$ induces an injection $\mathrm{Neg}(X) \to \mathrm{Neg}(X)$. Assume $f(C) = f(C')$ for a negative curve $C$ and an integral (but not necessarily negative) curve $C'$. Then $f_*(C') = \alpha f_*(C)$ for some rational number $\alpha > 0$. Hence $C' - \alpha C \equiv 0$ by the injectivity of $f_*$. Then $C \cdot C' = \alpha C^2 < 0$, implying $C = C'$. In particular, $f(C)$ is a negative curve, since for any integral component $D$ in $f^*(f(C))$, we have $f(D) = f(C)$, hence $D = C$, and therefore

$$f(C)^2 = \frac{1}{\deg f}(f^*C)^2 = \frac{1}{\deg f}(mC)^2 < 0$$

for some positive integer $m$.
Next, we show that if $C$ is a negative curve, then $f^k(C)$ lies in the ramification locus $R_f$ of $f$ for infinitely many integers $k$. Assuming the contrary, we have $f^*\left(f^{k+1}(C)\right) = f^k(C)$ for all but finitely many $k$. Set $b_k$ to be the integer with $f^*(f^{k+1}(C)) = b_k f^k(C)$, so $b_k$ is 1 for all but finitely many $k$. and hence

$$C^2 = \frac{\deg f}{b_0^2} \cdot f(C)^2 = \frac{(\deg f)^2}{b_0^2 b_1^2} \cdot f^2(C)^2 = \cdots \in \bigcap_{k=0}^{\infty} \frac{(\deg f)^k}{\prod_{i=0}^{k} b_i^2} \cdot \mathbb{Z} = \emptyset,$$

which is a contradiction. Hence $f^*\left(f^{k+1}(C)\right) = b_k \cdot f^k(C)$ for infinitely many $k$ with $b_k > 1$, implying $f^k(C) \subseteq \operatorname{supp} R_f$.
To conclude, let $T(X)$ be the subset of $\mathrm{Neg}(X)$ containing the negative curves with support in the ramification divisor. Note that $f$ is an injection on $\mathrm{Neg}(X)$, and $\operatorname{supp} R_f$ contains only finitely many components, hence the set $T(X)$ is finite. Let $C$ be a curve in $T(X)$. There exist infinitely many integers $k$ such that $f^k(C) \in T(X)$. Therefore, there exist two integers $k_1, k_2$ such that $f^{k_1}(C) = f^{k_2}(C)$. Hence $f^{|k_1 - k_2|}(C) = C$ by the injectivity of $f$. We take $k_C$ to be the smallest integer such that $f^{k_C}(C) = C$, and define $k_0 := \prod_{C \in T(X)} k_C$. We see that $f^{k_0}$ is the identity on $T(X)$. Then, we can apply Lemma 4.2 to $\mathrm{Neg}(X), T(X)$ and $f$, and deduce that $\mathrm{Neg}(X)$ is a finite set. □

## 4.2 Classification of surfaces admitting a non-trivial surjective separable endomorphism

Given the result that a surface admitting a non-trivial surjective separable endomorphism has only finitely many negative curves, it is then natural to ask: *When does a surface admit a non-trivial surjective separable endomorphism?* We have a complete classification in characteristic 0 and we know some essential conditions for a surface in characteristic $p$ to have such an endomorphism. As a byproduct, we get a better description of the set of negative curves on the surface.
First, we show that the proof of Proposition 4.3 can be generalized, showing that every negative curve on a surface admitting a non-trivial surjective separable endomorphism lies in the ramification locus of a power of the endomorphism.

**Proposition 4.4.** *Let $X$ be a surface having a non-trivial surjective separable endomorphism $f : X \to X$. Then, any negative curve $C$ is in the ramification divisor $R_{f^m}$ for some $m$.*

*Proof.* Since $f$ induces an injection on $\mathrm{Neg}(X)$ and $\mathrm{Neg}(X)$ is a finite set, there exists an integer $j$ such that $f^m$ induces the identity on $\mathrm{Neg}(X)$. Hence $f^{mn}$ is the identity on $\mathrm{Neg}(X)$ for all positive integers $n$. By the proof of Proposition 4.3, the curve $f^{mn}(C)$ is in $\operatorname{supp} R_{f^m}$ for some $n$, hence $C = f^{mn}(C) \subseteq \operatorname{supp} R_{f^m}$ for some $n$. □

### 4.2.1 Case of $\kappa(X) \geq 0$

If a surface $X$ with $\kappa(X) \geq 0$ admits a non-trivial surjective separable endomorphism, we can show that such an endomorphism is finite étale, and $X$ is a minimal surface. As a corollary, there are no negative curves on $X$ at all.

**Proposition 4.5** ([Fuj02, Lemma 2.3])**.** *Let $X$ be a surface with $\kappa(X) \geq 0$. If $X$ admits a non-trivial surjective endomorphism $f : X \to X$, then $f$ is finite étale, $X$ is minimal and has no negative curves.*

*Proof.* By Lemma 4.1, $f$ is finite. We claim that $f$ is étale. Taking the ramification formula $K_X \sim f^*K_X + R$ and iterating it, we get

$$K_X \sim (f^*)^n K_X + (f^*)^{n-1} R + \cdots + f^*R + R.$$

If $R \neq 0$, we take an ample divisor $H$ and see that $(f^*)^k R \cdot H > 0$, and $(f^*)^k K_X \cdot H \geq 0$ as $K_X$ is $\mathbb{Q}$-effective. We can derive that

$$K_X \cdot H = ((f^*)^n K_X + (f^*)^{n-1} R + \cdots + f^*R + R) \cdot H$$

tends to infinity if $n$ tends to infinity, a contradiction. Hence $R = 0$ and $f$ is étale.

If there exists a negative curve on $X$, it has to lie in the ramification locus of $f$, but $R = 0$. So there is no negative curve on $X$. □

Having these properties of $X$ and $f$, we can classify all surfaces with non-negative Kodaira dimension admitting a non-trivial surjective separable endomorphism.

**Theorem 4.6** ([Nak10, Theorem 1.2])**.** *Let $X$ be a surface with $\kappa(X) \geq 0$ admitting a nontrivial surjective separable endomorphism. Then $X$ is minimal and $\chi(X, \mathcal{O}_X) = 0$. Moreover,*

*1) If $\kappa(X) = 0$, then $X$ is either a hyperelliptic surface, a quasi-hyperelliptic surface or an abelian surface.*

*2) If $\kappa(X) = 1$, then the Iitaka fibration $\phi : X \to C$ gives $X$ the unique structure of an elliptic surface over a curve $C$.*

*Proof.* By Proposition 4.5, $X$ is minimal. We see by the Hirzebruch-Riemann-Roch theorem that $\chi(X, \mathcal{O}_X) = \deg f \cdot \chi(X, \mathcal{O}_X)$, hence $\chi(X, \mathcal{O}_X) = 0$.
Then, we exclude the case $\kappa(X) = 2$. Assume $X$ is of general type. Then since $f$ is finite étale by Proposition 4.5, we have $K_X \sim f^*K_X$ and hence $f$ induces an isomorphism on $H^0(X, mK_X)$. Let $\phi : X \dashrightarrow Y$ be the Iitaka fibration, then there is an automorphism $h$ of $Y$ induced by $f$ making the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ Y & \xrightarrow[h]{} & Y. \end{array}$$

Since $\phi$ is birational, $f$ has to be birational as well, hence $f$ is an isomorphism since it is also finite étale. This contradicts the assumption that $f$ is not an isomorphism.
Now by the Enriques-Kodaira classification of surfaces, if $\kappa(X) = 0$, then $X$ is a hyperelliptic surface or a quasi-hyperelliptic surface or an abelian surface; if $\kappa(X) = 1$, then $X$ is an elliptic surface or a quasi-elliptic surface.
Finally, we exclude the possibility that $X$ is a quasi-elliptic surface with $\kappa(X) = 1$. Taking the Iitaka fibration $\phi : X \to C$, there is an automorphism $h$ of $C$ induced by $f$ making the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} \\ C & \xrightarrow[h]{} & C. \end{array}$$

Take a general point $c \in C$ such that the fibres $\phi^{-1}(c)$ and $\phi^{-1}(h(c))$ are irreducible reduced curves of geometric genus 1, and each curve has only one ordinary cusp. Then, $f$ induces a finite étale morphism $\phi^{-1}(c) \to \phi^{-1}(h(c))$. Pulling back this morphism along the normalization $\mathbb{P}^1 \to \phi^{-1}(h(c))$, we get a non-trivial finite étale covering of $\mathbb{P}^1$, which is impossible. $\square$

**Remark 4.7.** The converse direction of the preceding theorem also holds, i.e. if a surface belongs to one of the types of surfaces mentioned above, then it admits a non-trivial surjective separable endomorphism. See [Nak10] for a proof.

#### 4.2.2 Case of $\kappa(X) = -\infty$

Next we turn to the case where $\kappa(X) = -\infty$. Then $X$ is either rational or a ruled surface by the Enriques-Kodaira classification of surfaces. We have the following characterization of surfaces with $\kappa(X) = -\infty$ admitting a non-trivial surjective endomorphism. However the proof is too long to be given here. We refer to [Nak02] and [Nak10] for a complete proof.

**Theorem 4.8** ([Nak02, Theorem 3 & Theorem 17]; [Nak10, Theorem 1.1])**.** *Let $X$ be a surface with Kodaira dimension $\kappa(X) = -\infty$. Assume that $X$ admits a non-trivial surjective separable endomorphism. Define the irregularity $q(X) = \dim \mathrm{Alb}(X)$. Then $X$ is one of the following surfaces:*

*1) $q(X) = 0$, and $X$ is rational with at most finitely many negative curves and $-K_X$ is big. Moreover, if the characteristic of the ground field is $0$, then $X$ is a toric surface.*

*2) $q(X) \geq 1$, and $X$ is a $\mathbb{P}^1$-bundle over a smooth projective curve $T$ of genus $q(X)$.*

**Remark 4.9.** The converse direction of the preceeding theorem holds in characteristic 0. See [Nak02] for a proof. It is not clear whether the converse direction holds in characteristic $p$.

Using the classifications, we can give an explicit description of negative curves on surfaces with $\kappa(X) = -\infty$ admitting a non-trivial surjective separable endomorphism. In the case where $\kappa(X) = -\infty$, the surface $X$ is either a rational surface where $-K_X$ is big, or a $\mathbb{P}^1$-bundle. We discuss the lower bound of self intersection of curves in each case.

- Case 1: $X$ is rational and $-K_X$ is big. In this case $-nK_X$ is effective for some $n \in \mathbb{N}$. By Proposition 3.2, a negative curve is either a component of the negative part of the Zariski decomposition of $-nK_X$, or has self intersection larger equal than $-2$. Moreover, if $X$ is over a field of characteristic 0, then $X$ is toric. In this case, the Picard group $\mathrm{Pic}(X)$ is finitely generated, and the generators of $\mathrm{Pic}(X)$ are precisely the generators of the homogeneous coordinate ring of $X$. See [Cox95] for a proof. If a curve on $X$ has negative self intersection, it has to be a generator of $\mathrm{Pic}(X)$. Hence the bound is given by the minimal self intersection of the generators of the homogeneous coordinate ring.

- Case 2: $X$ is a $\mathbb{P}^1$-bundle. The surface $X$ has Picard number $\rho(X) = 2$, since the Picard group is generated by a section $C_0$ which has the minimal self intersection among all sections, and a fibre $f$ of the fibration (See [Har77, Proposition V.2.3]). Hence $X$ has $C_0$ as the unique negative curve if $C_0^2 < 0$, or $X$ has no negative curves at all if $C_0^2 \geq 0$. Moreover, we can write $X$ as as the projective bundle $\mathbb{P}(\mathcal{E})$ over some curve $C$, where $\mathcal{E}$ is a locally free sheaf of rank 2 on $C$. It is possible to find a $\mathcal{E}$ such that $X \cong \mathbb{P}(\mathcal{E})$, and $H^0(C, \mathcal{E}) \neq 0$ but $H^0(C, \mathcal{E} \otimes \mathcal{L}) = 0$ for all line bundles $\mathcal{L}$ on $C$ with $\deg \mathcal{L} < 0$ (See [Har77, Proposition V.2.8]). If $\mathcal{E}$ satisfies the assumptions above, the number $\deg(\det \mathcal{E})$ is uniquely determined and $C_0^2 = \deg(\det \mathcal{E})$ (See [Har77, Proposition V.2.9]).

We summarize the information of negative curves as a table below.

| $\kappa(X)$ | Classification | Negative curves |
|---|---|---|
| 0 | $\chi(X, \mathcal{O}_X) = 0$ | None |
| $-\infty$ | rational, $-K_X$ big | Contained in the negative part of $-K_X$ or has self intersection $\geq -2$ |
| | (if also toric) | Generators of the homogeneous coordinate ring |
| | $\mathbb{P}^1$-bundle | At most one |

# 5 Rational surfaces of positive characteristic with unbounded negativity

In this chapter we answer Question 2 in the Introduction, which asks for counterexamples in positive characteristic with Kodaira dimension less than 2. We show that in characteristic $p$, there exists a blow up of $\mathbb{P}^2$ on which the Bounded Negativity Conjecture fails. As corollaries, we deduce that for every surface $X$ in characteristic $p$, there exists a smooth blow up of $X$ on which the Bounded Negativity Conjecture fails, and bounded negativity is not stable under finite pullback and blow up. In this chapter we work over an algebraically closed field $k$ of characteristic $p$.

## 5.1 Rational surfaces with unbounded negativity

We follow [CdB21] and construct a rational surface with unbounded negativity.

**Theorem 5.1** ([CdB21, Main Theorem]). *Let $m, d$ be positive integers such that $dm = p^e - 1$ for some positive integer $e$. Define*

$$Z_m := \left\{ [x_0 : x_1 : x_2] \in \mathbb{P}^2 | x_0^m = x_1^m = x_2^m \right\}.$$

*Let $R_m$ be the blow up of $\mathbb{P}^2$ along $Z_m$. Write $C_d$ for the image of $C_1 := V(x_0 + x_1 + x_2)$ under the d-power map $[x_0 : x_1 : x_2] \mapsto [x_0^d : x_1^d : x_2^d]$. Then the strict transform $\widetilde{C_d}$ in $R_m$ is a smooth rational curve and has self intersection $d(3 - m) - 1$. In particular, if $m > 3$, the surface $R_m$ does not have bounded negativity.*

To prove the theorem we first need several lemmas. First note that $Z_m$ is defined by the equations $s_0 = x_1^m - x_2^m, s_1 = x_2^m - x_0^m, s_2 = x_0^m - x_1^m$. The preimage of $Z_m$ in $R_m$ is an effective Cartier divisor cut out by the pullbacks of the $s_i$. Hence they define an embedding $R_m \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2$.

**Lemma 5.2** ([CdB21, Lemma 1.1]). *$R_m$ is the complete intersection of $y_0 + y_1 + y_2 = 0$ and $x_0^m y_0 + x_1^m y_1 + x_2^m y_2 = 0$ in $\mathbb{P}^2 \times \mathbb{P}^2$ with coordinates $[x_0 : x_1 : x_2]$ and $[y_0 : y_1 : y_2]$. In particular, the canonical divisor of $R_m$ is $K_{R_m} = \mathcal{O}_{R_m}(m - 3, -1)$.*

*Proof.* By definition of the blow up, $R_m$ is the vanishing locus

$$V \begin{pmatrix} y_0(x_2^m - x_0^m) - y_1(x_1^m - x_2^m) \\ y_1(x_0^m - x_1^m) - y_2(x_2^m - x_0^m) \\ y_2(x_1^m - x_2^m) - y_0(x_0^m - x_1^m) \end{pmatrix}.$$

The equation $s_0+s_1+s_2 = 0$ implies that $R_m$ is contained in the locus defined by the equation $y_0 + y_1 + y_2 = 0$. The first equation $y_0(x_2^m - x_0^m) = y_1(x_1^m - x_2^m)$ can be rewritten as $(y_0 + y_1)x_2^m = x_0^m y_0 + x_1^m y_1$, which is equivalent to $x_0^m y_0 + x_1^m y_1 + x_2^m y_2 = 0$. The other two equations follow by symmetry.
Now, by the adjunction formula, we have $K_{R_m} = \mathcal{O}_{R_m}(m - 3, -1)$, as $K_{\mathbb{P}^2 \times \mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(-3, -3)$. □

**Lemma 5.3** ([CdB21, Lemma 1.2]). *If $dm = p^e - 1$ for some positive integer $e$, the map $\widetilde{\phi_d} : C_1 \to \mathbb{P}^2 \times \mathbb{P}^2$ given by*

$$[x_0 : x_1 : x_2] \mapsto \Big( [x_0^d : x_1^d : x_2^d], [x_0 : x_1 : x_2] \Big)$$

*factors through $R_m$. The map is a closed immersion and the image is a smooth rational curve which coincides with the strict transform of $C_d$ in $R_m$.*

*Proof.* The equation $x_0 + x_1 + x_2 = 0$ implies that $\widetilde{\phi_d}(C_1)$ lies in the locus $y_0 + y_1 + y_2 = 0$. The equation $x_0^m y_0 + x_1^m y_1 + x_2^m y_2 = 0$ pulls back

along $\widetilde{\phi_d}$ to $x_0^{p^e}+x_1^{p^e}+x_2^{p^e}=0$, hence vanishes on $V(x_0+x_1+x_2)$ since $x_0^{p^e}+x_1^{p^e}+x_2^{p^e}=(x_0+x_1+x_2)^{p^e}$. Therefore, $\widetilde{\phi_d}(C_1)$ factors through $R_m$. Since the map is the identity on the second factor of $\mathbb{P}^2\times\mathbb{P}^2$, it is a closed immersion. It is smooth as the derivatives of its defining regular functions are $\left([dx_0^{d-1}:dx_1^{d-1}:dx_2^{d-1}],[1:1:1]\right)$ and vanish nowhere, since $d$ is not divisible by $p$. In particular, $\widetilde{\phi_d}(C_1)$ is isomorphic to $C_1$, hence rational.
Via projection onto the first factor of $\mathbb{P}^2\times\mathbb{P}^2$ we see that the curve $\widetilde{\phi_d}(C_1)$ maps to $C_d$. Since $\widetilde{\phi_d}(C_1)$ is integral, it follows that it is the strict transform of $C_d$. □

*Proof of Theorem 5.1.* By the explicit form of the closed immersions $C_1\hookrightarrow R_m\hookrightarrow\mathbb{P}^2\times\mathbb{P}^2$, we have $\widetilde{\phi_d}^*\mathcal{O}_{R_m}(a,b)=\mathcal{O}_{C_1}(da+b)$. So we get

$$K_{R_m}\cdot\widetilde{C_d}=\deg\mathcal{O}_{R_m}(m-3,-1)|_{\widetilde{C_d}}=d(m-3)-1.$$

Then by the adjunction formula,

$$\widetilde{C_d}^2=2g(\widetilde{C_d})-2-K_{R_m}\cdot\widetilde{C_d}=d(3-m)-1.$$

□

Theorem 5.1 also shows that bounded negativity is not stable under finite pullbacks. We note that $R_m$ fits into a Cartesian diagram:

$$\begin{array}{ccc} R_m & \longrightarrow & \mathrm{Bl}_{[1:1:1]}\mathbb{P}^2 \\ \downarrow & & \downarrow \\ \mathbb{P}^2 & \xrightarrow{[X_0:X_1:X_2]\mapsto[X_0^m:X_1^m:X_2^m]} & \mathbb{P}^2. \end{array}$$

The upper morphism $R_m\to\mathrm{Bl}_{[1:1:1]}\mathbb{P}^2$ is finite, and $\mathrm{Bl}_{[1:1:1]}\mathbb{P}^2$ has bounded negativity.

**Corollary 5.4.** *Having bounded negativity is not a property stable under finite pullbacks in characteristic $p$.*

Since every surface $X$ admits a finite morphism to $\mathbb{P}^2$ by Proposition 2.5, we can pull back the blow up $R_m$ to a blow up of $X$ which does not have the bounded negativity.

**Proposition 5.5.** *Let $X$ be a surface. There exists a smooth blow up of $X$ at finitely many points which does not have bounded negativity.*

*Proof.* Take a finite morphism $f:X\to\mathbb{P}^2$ as in Proposition 2.5. There are two possible cases:
Case 1: The points $Z_m$ do not intersect with the branch locus $B$ of $f$. In

this case, the preimage $f^{-1}(Z_m)$ consists only of reduced points, hence the blow up $\widetilde{X} := \mathrm{Bl}_{f^{-1}(Z_m)}X$ is smooth and fits into the Cartesian diagram:

$$\begin{array}{ccc} \widetilde{X} = \mathrm{Bl}_{f^{-1}(Z_m)}X & \longrightarrow & R_m = \mathrm{Bl}_{Z_m}\mathbb{P}^2 \\ \downarrow & & \downarrow \\ X & \xrightarrow{\quad f \quad} & \mathbb{P}^2. \end{array}$$

The morphism $\widetilde{X} \to R_m$ is finite and $R_m$ does not have bounded negativity, hence by Proposition 3.3 the surface $\widetilde{X}$ does not have bounded negativity.
Case 2: There are points in $Z_m$ lying on the branch locus $B$. We show that in this case we can find an automorphism $g$ of $\mathbb{P}^2$, mapping $Z_m$ away from $B$. Blowing up the points in $g(Z_m)$ gives a rational surface isomorphic to $R_m$ and we can reduce to the first case. Let $a_1, \dots, a_{m^2}$ be the points in $Z_m$ and define

$$G_i := \{g \in \mathrm{PGL}_3 \mid g(a_i) \in B\}.$$

Then, $G_i$ is a proper Zariski closed subset in the group $\mathrm{PGL}_3$ endowed with Zariski topology. Hence $\bigcap_i \mathrm{PGL}_3 \backslash G_i = \mathrm{PGL}_3 \backslash \bigcup_i G_i$ is a dense open subset, the points in which act on $\mathbb{P}^2$ via automorphisms sending the points $Z_m$ away from $B$. ☐

**Corollary 5.6.** *Having bounded negativity is not a property stable under blow ups in characteristic $p$.*

## 5.2 Relations with the Frobenius morphism

We then consider the rational surfaces of type $R_m$ in a more generalized way. It will turn out that the curves in Proposition 1.1 can be regarded as strict transforms of curves in Theorem 5.1. We follow here [CdB21].

**Definition 5.7.** Let $m, n$ be positive integers that are not divisible by $p$, and let $r$ be a non-negative integer. We write $R_{m,n,r}$ for the normal surface

$$R_{m,n,r} := \left\{ ([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \in \mathbb{P}^2 \times \mathbb{P}^2 \;\middle|\; \begin{array}{r} y_0^n + y_1^n + y_2^n = 0 \\ x_0^m y_0^r + x_1^m y_1^r + x_2^m y_2^r = 0 \end{array} \right\}.$$

Moreover, we write $X_n$ for the Fermat curve $x_0^n + x_1^n + x_2^n = 0$ in $\mathbb{P}^2$.

Note then $R_{m,1,1}$ is just $R_m$, and $R_{m,n,0}$ is isomorphic to $X_m \times X_n$.

**Lemma 5.8** ([CdB21, Section 3.1])**.** *The surface $R_{m,n,r}$ is smooth if and only if $m = 1$ or $r \in 0, 1$. Otherwise it has the singular locus $V(x_0y_0, x_1y_1, x_2y_2)$, which consists of $3n$ singular points:*

$$\{([1:0:0],[0:s:t]), ([0:1:0],[s:0:t]), ([0:0:1],[s:t:0]) \mid s^n + t^n = 0\}$$

*Proof.* If $r = 0$, the surface $R_{m,n,r}$ is a product of two smooth curves. Thus we assume $r \geq 1$. Take the Jacobian matrix of the defining equations:

$$\begin{pmatrix} 0 & 0 & 0 & ny_0^{n-1} & ny_1^{n-1} & ny_2^{n-2} \\ mx_0^{m-1}y_0^r & mx_1^{m-1}y_1^r & mx_2^{m-1}y_2^r & rx_0^m y_0^{r-1} & rx_1^m y_1^{r-1} & rx_2^m y_2^{r-1} \end{pmatrix}.$$

The two rows are linearly independent unless the second row is 0. If $m = 1$, the second row is 0 if and only if $y_0 = y_1 = y_2 = 0$, which is not possible, hence $R_{1,n,r}$ is smooth. If $m > 1$, the second row is 0 if and only if $x_0y_0 = x_1y_1 = x_2y_2 = 0$, which yields the description of the singular locus. □

There is also a projection $p_2 : R_{m,n,r} \to X_n$ from the surface onto its second factor.

**Lemma 5.9** ([CdB21, Section 3.1])**.** *$p_2$ is smooth away from $V(y_0y_1y_2) \subseteq X_n$. Every singular fibre of $p_2$ consists of $m$ lines meeting at a point.*

*Proof.* The image of the singular locus of $R_{m,n,r}$ lies in $V(y_0y_1y_2)$. Over $X_n \backslash V(y_0y_1y_2)$ every fibre is isomorphic to $X_m$, hence $p_2$ is smooth away from $V(y_0y_1y_2)$.
To describe the singular fibres over $V(y_0y_1y_2)$, it suffices to calculate the singular fibres over $V(y_0)$. The rest follows from the symmetry of the equations. The fibre over $[0 : s : t]$ is cut out by the equation $s^r x_1^m + t^r x_2^m = 0$. The equation cuts out $m$ lines meeting at $[1 : 0 : 0]$, since $s, t \neq 0$ and $p$ does not divide $m$. □

**Proposition 5.10** ([CdB21, Section 3.2])**.** *For positive integers $a, b$ not divisible by $p$, we define the finite morphism*

$$\pi_{a,b} : R_{am,bn,br} \to R_{m,n,r}$$
$$([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \mapsto \left([x_0^a : x_1^a : x_2^a], [y_0^b : y_1^b : y_2^b]\right).$$

*If $a = 1$, we have a Cartesian diagram:*

$$\begin{array}{ccc} R_{m,bn,br} & \xrightarrow{\pi_{1,b}} & R_{m,n,r} \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_2} \\ X_{bn} & \longrightarrow & X_n. \end{array}$$

*Proof.* Explicit computation. □

**Lemma 5.11** ([CdB21, Lemma 3.3])**.** *Let $m, n$ be positive integers not divisible by $p$, let $r$ be a non-negative integer ,and let $a$ be an integer such that $r + am \geq 0$. Then, the rational map*

$$\psi_a : \mathbb{P}^2 \times \mathbb{P}^2 \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^2$$
$$([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \longmapsto ([x_0y_0^a : x_1y_1^a : x_2y_2^a], [y_0 : y_1 : y_2])$$

*maps $R_{m,n,r+am}$ birationally onto $R_{m,n,r}$.*

*Proof.* Computation in coordinates shows that the image of $R_{m,n,r+am}$ is contained in $R_{m,n,r}$. The map is birational since it has a rational inverse $\psi_{-a}$. The base locus of $\psi_a$ and $\psi_{-a}$ is $V(y_0y_1y_2)$ which does not contain the two surfaces. □

We now clarify the relation between Theorem 5.1 and Poposition 1.1. Recall that $\widetilde{C_d}$ is a curve in $R_m \cong R_{m,1,1}$ and $\pi_{1,m}$ is a finite morphism $R_{m,m,m} \to R_{m,1,1}$.

**Corollary 5.12** ([CdB21, Corollary 3.4 & Section 3.5])**.** *The rational map $\psi_{-1}$ maps $X_m \times X_m \cong R_{m,m,0}$ birationally onto $R_{m,m,m}$. Moreover, if $d$ is a positive integer such that $dm = p^e - 1$ for some positive integer $e$, then the strict transform of $\pi_{1,m}^*\widetilde{C_d}$ under $\psi_{-1}$ is the transpose $\Gamma_{F^e}^\intercal$ of the graph of the $p^e$-power Frobenius map on $X_m$.*

*Proof.* The first statement follows from Lemma 5.11. Now for the second statement, recall that the transpose $\Gamma_{F^e}^\intercal$ is given by a section $s$ of $p_2$:

$$\begin{aligned} s : X_m &\to X_m \times X_m \\ [y_0 : y_1 : y_2] &\mapsto \left([y_0^{p^e} : y_1^{p^e} : y_2^{p^e}], [y_0 : y_1 : y_2]\right). \end{aligned}$$

Furthermore, recall that by Lemma 5.3, the curve $\widetilde{C_d}$ is given by the image of

$$\begin{aligned} \phi_d : X_1 &\to R_{m,1,1} \\ [y_0 : y_1 : y_2] &\mapsto \left([y_0^d : y_1^d : y_2^d], [y_0 : y_1 : y_2]\right). \end{aligned}$$

By Lemma 5.10 the map $\pi_{1,m}$ fits into a Cartesian diagram:

$$\begin{array}{ccc} R_{m,m,m} & \xrightarrow{\pi_{1,m}} & R_{m,1,1} \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_2} \\ X_m & \longrightarrow & X_1. \end{array}$$

Combining the diagram and $\phi_d$, we see that $\pi_{1,m}^*\widetilde{C_d}$ is the image of the section:

$$\begin{aligned} X_m &\to R_{m,m,m} \\ [y_0 : y_1 : y_2] &\mapsto \left([y_0^{dm} : y_1^{dm} : y_2^{dm}], [y_0 : y_1 : y_2]\right). \end{aligned}$$

In particular, $\pi_{1,m}^*\widetilde{C_d}$ it is a smooth irreducible reduced curve. Now the rational map $\psi_{-1} \circ s$ is given by

$$\begin{aligned} \psi_{-1} \circ s : X_m &\dashrightarrow R_{m,m,m} \\ [y_0 : y_1 : y_2] &\longmapsto \left(y_0^{p^e-1} : y_1^{p^e-1} : y_2^{p^e-1}], [y_0 : y_1 : y_2]\right). \end{aligned}$$

The image agrees with $\pi_{1,m}^*\widetilde{C_d}$ on the locus where the rational map is defined, hence $\Gamma_{F^e}^\intercal$ is the strict transform of $\pi_{1,m}^*\widetilde{C_d}$ under $\psi_{-1}$. $\square$

We can now give an alternative proof of Theorem 5.1.

*Proof of Theorem 5.1.* We first note that $\psi_{-1}$ corresponds to a sub linear system of the bundle $\mathcal{O}(1,2)$, since it can be written as

$$[x_0 : x_1 : x_2] \times [y_0 : y_1 : y_2] \mapsto [x_0y_1y_2 : x_1y_0y_2 : x_2y_0y_1] \times [y_0 : y_1 : y_2].$$

The sub linear system has base points $V(y_0y_1y_2)$, which contains the $3m^2$ points

$$\left\{ [0 : u : v] \times [0 : s : t], [u : 0 : v] \times [s : 0 : t], [u : v : 0] \times [s : t : 0] \;\middle|\; \begin{aligned} u^m + v^m &= 0 \\ s^m + t^m &= 0 \end{aligned} \right\}.$$

Blowing up these points, we get a surface $\widetilde{R_{m,m,0}}$ and a morphism $\widetilde{\psi_{-1}} : \widetilde{R_{m,m,0}} \to R_{m,m,m}$ ([Har77, Example II.7.17.3]) making the following diagram commute:

$$\begin{array}{ccc} \widetilde{R_{m,m,0}} & & \\ \downarrow \alpha & \searrow^{\widetilde{\psi_{-1}}} & \\ R_{m,m,0} & \overset{\psi_{-1}}{\dashrightarrow} & R_{m,m,m}. \end{array}$$

$\Gamma_{p^e}^\intercal$ passes through $3m$ points in $V(y_0y_1y_2)$. Hence

$$\begin{aligned} \left(\widetilde{C_d}\right)^2 &= \frac{1}{m^2}\left(\pi_{1,m}^*\widetilde{C_d}\right)^2 = \frac{1}{m^2}\left(\widetilde{\psi_{-1}}^*\pi_{1,m}^*\widetilde{C_d}\right)^2 \\ &= \frac{1}{m^2}\left(\alpha^*\Gamma_{F^e}^\intercal\right)^2 = \frac{1}{m^2}\left(\left(\Gamma_{F^e}^\intercal\right)^2 - 3m\right) \\ &= \frac{1}{m^2}\left(p^e(2-2g) - 3m\right) = \frac{1}{m^2}\left((dm+1)(2-(m-1)(m-2)) - 3m\right) \\ &= d(3-m) - 1. \end{aligned}$$

$\square$

# 6 Surfaces with infinitely many negative curves of fixed genus and self intersection

In this chapter we give a positive answer to Question 3 in the Introduction about the existence of surfaces having infinitely many negative curves of a fixed self intersection. We first show that given an integer $m \geq 1$, there exists a complex surface with infinitely many curves having self intersection $-m$.

**Theorem 6.1** ([BHK$^+$13, Theorem 4.1])**.** *Given an integer $m \geq 1$, there exists a complex surface with infinitely many negative curves with self intersection $-m$.*

*Proof.* Take an elliptic curve $E$ without complex multiplication and denote $A := E \times E$. Then $A$ is an abelian surface. Write $F_1, F_2$ for the fibres of the two projections onto $E$, and $\Delta$ for the diagonal. The Néron-Severi group of $A$ is generated by $F_1, F_2, \Delta$. By [BS08, Proposition 2.3], for a positive integer $n$, the class $n(n+1)F_1 + (n+1)F_2 - n\Delta$ is numerically equivalent to an elliptic curve on $A$. Write $E_n$ for the corresponding elliptic curve. Note that $E_n^2 = 0$. Using translations on $A$, we may assume that the origin of $E_n$ is the origin of $A$. In this case, $E_n$ forms a subgroup of $A$. Take a positive integer $t$ such that $t^2 \geq m$ and consider the $t$-torsion points on $E_n$. All the $t$-torsion points of all $E_n$ (there are $t^2$ points for each $E_n$) lie on the $t$-torsion points on $A$ (there are $t^4$ points). Since there are only finitely many $t$-torsion points on $A$, we can find an infinite subsequence of $(E_n)_n$ such that they pass through the same $t^2$ many $t$-torsion points on $A$. Blowing up $m$ points among the $t^2$ points, we get that the proper transform $C_n$ of $E_n$ in the subsequence has self intersection $C_n^2 = E_n^2 - m = -m$. $\square$

We may push the discussion even further. *Given two integers $m \geq 1, g \geq 0$, does there exist a surface $X$ with infinitely many genus $g$ curves having self intersection $-m$?* The answer is yes for $m \geq g/2+1$. Moreover, if we restrict to the complex numbers, we can also find such a surface for all $g \geq 0, m \geq 2$. For the case $g = 0, m = 1$ we have the following explicit example. This proposition can be found in [Nag60, Theorem 4a], but the proof there is written in classical language. We formulate here a proof in the language of modern algebraic geometry.

**Proposition 6.2.** *Let $C_1, C_2$ be two general smooth cubic curves on $\mathbb{P}^2$ having transversal intersections. Then the surface obtained by blowing up $\mathbb{P}^2$ at the nine intersecting points of $C_1, C_2$ is an elliptic surface with infinitely many $(-1)$-curves.*

*Proof.* Take the elliptic fibration given by the pencil generated by two general smooth cubic curves $C_1, C_2$ on $\mathbb{P}^2$. More precisely, we take two general functions in $\mathcal{O}(3)$ and consider the corresponding rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$. By blowing up the 9 base points we get an elliptic fibration whose almost all fibres are smooth elliptic curves. Write $e_i$ for the 9 intersection points of $C_1, C_2$ and $E_i$ for the 9 execptional divisors.
The anti-canonical divisor $-K_X$ is given by $\mathcal{O}(3) - \sum_i E_i$, which is also the strict transform of a cubic curve going through the nine base points. Hence the anti-canonical divisor is just the fibre class. In particular, $-K_X$ has intersection number 1 with a section. Let $C$ be a section of the elliptic fibration. By the adjunction formula:

$$C^2 = \deg K_X|_C - K_X \cdot E = -1.$$

Pick the point $e_1$ as the neutral element of $C_1$ and the exceptional divisor $E_1$ as the neutral element of the generic fibre. Since $C_1, C_2$ are chosen very generally, we can assume $e_2$ is not a torsion point of $C_1$. This implies that $E_2$ is not a torsion in the generic fibre, since the group structure of the generic fibre is compatible with $C_1$. By taking all of the multiples of $E_2$, we get infinitely many sections and hence infinitely many $(-1)$-curves. □

To prove the case where $g \geq 0, m \geq 2$, we need the Hurwitz covering which gives a finite morphism from a smooth curve of genus $g$ to $\mathbb{P}^1$ of degree $m$.

**Proposition 6.3.** *For any pair of integers $g, m$ satisfying $m \geq g/2+1$, every smooth curve of genus $g$ has a finite morphism to $\mathbb{P}^1$ of degree $m$. Moreover, if the ground field is the complex numbers, then for any $g \geq 0, 2 \leq m \leq g/2+1$, there exists a finite morphism $f : C \to \mathbb{P}^1_{\mathbb{C}}$ of degree $m$, where $C$ is a smooth complex curve of genus $g$.*

*Proof.* The case $m \geq g/2+1$ follows from [KL74, Theorem 5], and the case $m \leq g/2+1$ over the complex numbers follows from [ACG11, Theorem 12.3]. We prove here only the special case $m \geq 2g$ for arbitrary characteristic. In this case, we pick a divisor $D$ on $C$ with degree $m$. By [Har77, Corollary IV.3.2] , $D$ is base point free. By Proposition 2.5, we get a finite morphism $g : C \to \mathbb{P}^1$ satisfying $\mathcal{O}(D) \cong g^*\mathcal{O}(1)$. It follows that $\deg g = \deg D = m$. □

**Theorem 6.4.** *Given two integers $g \geq 0, m \geq g/2+1$, there exists a surface with infinitely many negative curves of genus $g$ and self intersection $-m$.*

*Proof.* Let $f : X \to \mathbb{P}^1$ be the elliptic fibration given in Proposition 6.2. Take a degree $m$ morphism $f' : C \to \mathbb{P}^1$ where $C$ is a smooth curve of genus $g$, whose existence is ensured by Proposition 6.3 and consider the following Cartesian diagram:

$$\begin{array}{ccc} X \times_{\mathbb{P}^1} C & \longrightarrow & X \\ \downarrow & & \downarrow f \\ C & \xrightarrow{f'} & \mathbb{P}^1. \end{array}$$

By Proposition 6.2, every section of $f$ gives a $(-1)$-curve. The sections are pulled back to sections of $X \times_{\mathbb{P}^1} C \to C$ and have self intersection $-m$. □

**Theorem 6.5** ([BHK+13] Theorem 4.3)**.** *Given two integers $m \geq 2, g \geq 0$, there exists a complex surface with infinitely many genus $g$ negative curves with self intersection $-m$.*

*Proof.* The proof is the same as Theorem 6.4, but over the complex numbers, Proposition 6.3 also ensures the existence of degree $m$ cover $C \to \mathbb{P}^1$ when $2 \leq m \leq g/2+1$. □

**Question 6.6.** *Do there exist (complex) surfaces containing infinitely many negative curves of genus $g \geq 1$ and self intersection $-1$?*

# 7 Possible generalizations

We now present some branching results and possible adjustions of the bounded negativity conjecture.

## 7.1 Weak bounded negativity conjecture

We can strengthen our hypothesis on curves by bounding their geometric genus.

**Conjecture 7.1** (Weak Bounded Negativity Conjecture)**.** *For each smooth projective surface $X$ and each integer $g$, there exists a number $b(X, g)$ depending on $X, g$, such that $C^2 \geq -b(X, g)$ for all irreducible reduced curves $C$ of geometric genus $g(C) \leq g$.*

### 7.1.1 Case of characteristic 0

In the case of characteristic 0, [BBC$^+$12] proved the Weak Bounded Negativity Conjecture for surfaces with non-negative Kodaira dimension (See [BBC$^+$12, Proposition 3.5.3]). Hao found that one can modify the proof using the generalized Miyaoka-Yau inequality and gave the proof for arbitrary surfaces. Here, we follow Hao's paper [Hao19].

**Theorem 7.2** ([Hao19])**.** *Let $X$ be a complex surface. Then there exists a number $b(X, g)$ depending on $X$ and a non-negative integer $g$, such that for every irreducible reduced curve $C$ on $X$ with geometric genus $g$, we have the inequality $C^2 \geq -b(X, g)$.*

We separate the proof into three cases:

1) $-K_X$ effective;

2) $h^0(X, -K_X) = 0$ and $h^0(X, 2K_X + 2C)) = 0$;

3) $h^0(X, -K_X) = 0$ and $h^0(X, 2K_X + 2C) > 0$.

In the first case the bounded negativity follows from Proposition 3.2 and the bound is independent of $g(C)$ and $X$. The following result gives a bound for the second case which is also independent of $g(C)$.

**Proposition 7.3** ([Hao19, Lemma 1.3])**.** *Let $X$ be a surface such that $H^0(X, -K_X) = 0$. Let $C$ be an irreducible reduced curve on $X$ such that $H^0(X, 2K_X + 2C) = 0$. Then $C^2 \geq K_X^2 + \chi(X, \mathcal{O}_X) - 3$.*

*Proof.* We have a short exact sequence:

$$0 \to \mathcal{O}_X(2K_X + C) \to \mathcal{O}_X(2K_X + 2C) \to \mathcal{O}_C(2K_X + 2C) \to 0.$$

Since $h^0(X, 2K_X + 2C) = 0$, we see that $h^0(X, 2K_X + C) = 0$. Similarly, $h^0(X, -K_X - C) = 0$. Now by the Riemann-Roch theorem and the Serre duality,

$$\begin{aligned}
&h^0(X, 2K_X + C) + h^0(X, -K_X - C) - h^1(X, 2K_X + C) \\
&= h^0(X, 2K_X + C) - h^1(X, 2K_X + C) + h^2(X, 2K_X + C) \\
&= \chi(X, \mathcal{O}_X) + K_X^2 - C^2 + 3p_a(C) - 3.
\end{aligned}$$

Hence $C^2 \geq K_X^2 + \chi(X, \mathcal{O}_X) - 3$. □

For the third case, we need the following Proposition which follows directly from the logarithmic Miyaoka-Yau inequality.

**Proposition 7.4** ([Hao19, Corollary 1.8]). *Let $X$ be a complex surface such that $H^0(X, -K_X) = 0$. Let $C$ be a smooth irreducible reduced curve on $X$ such that $H^0(X, 2K_X + 2C) > 0$. Then $C^2 \geq K_X^2 - 3c_2(X) + 2 - 2g(C)$.*

*Proof.* By the logarithmic Miyaoka-Yau inequality (See Theorem 2.4),

$$(K_X + C)^2 \leq 3c_2(X) + 6g(C) - 6.$$

By the adjunction formula, we have $K_X \cdot C = 2g(C) - 2 - C^2$ and hence $(K_X + C)^2 = K_X^2 - C^2 + 4g(C) - 4$. Combining it with the inequality above, we get the result. □

**Theorem 7.5** ([Hao19, Theorem 1.9]). *Let $X$ be a surface such that $H^0(X, -K_X) = 0$. Let $C$ be an irreducible reduced curve on $X$ such that $H^0(X, 2K_X + 2C) > 0$. Then*

$$C^2 \geq \min \left\{ K_X^2 + \chi(X, \mathcal{O}_X) - 3, K_X^2 - 3c_2(X) + 2 - 2g(C) \right\}.$$

*Proof.* First define

$$\begin{aligned}
M(X, g) &:= K_X^2 + \chi(X, \mathcal{O}_X) - 3, \\
N(X, g) &:= K_X^2 - 3c_2(X) + 2 - 2g.
\end{aligned}$$

We know that the statement $C^2 \geq \min \{M(X, g), N(X, g)\}$ is true for $C$ smooth with genus $g$ by Proposition 7.4. Let $\widetilde{X}$ be the blow up of $X$ at a point $p$ and let $\widetilde{C}$ be the strict transform of $C$. If we can show that the inequality $\widetilde{C}^2 \geq \min \left\{ M(\widetilde{X}, g), N(\widetilde{X}, g) \right\}$ implies $C^2 \geq \min \{M(X, g), N(X, g)\}$, then we can blow up all the singularities of $C$ step by step and reduce to the smooth case in which the statement holds. We have the relations $M(\widetilde{X}, g) = M(X, g) - 1$ and $N(\widetilde{X}, g) = N(X, g) - 4$, and hence

$$\min\left\{M(\widetilde{X}, g), N(\widetilde{X}, g)\right\} \geq \min\{M(X, g), N(X, g)\} - 4.$$

Let $m$ be the multiplicity of $C$ at $p$. One has

$$C^2 - m^2 = \widetilde{C}^2 \geq \min\left\{M(\widetilde{X}, g), N(\widetilde{X}, g)\right\} \geq \min\{M(X, g), N(X, g)\} - 4.$$

Since $m \geq 2$ if $C$ is singular at $p$, the result follows. □

Now, Theorem 7.2 follows by combining Proposition 3.2, Proposition 7.3 and Theorem 7.5.

#### 7.1.2 Case of characteristic $p$

In characteristic $p$, the conjecture is false due to Proposition 1.1. Moreover, Theorem 5.1 gives a counterexample for rational surfaces since every $\widetilde{C_d}$ is smooth rational. The failure is due to the failure of the logarithmic Miyaoka-Yau inequality for $\mathbb{Q}$-effective logarithmic canonical divisors, as we are going to present now. We follow here Chapter 4 in [CdB21]. Recall the definitions of $R_m, C_d$ and $\widetilde{C_d}$ from Theorem 5.1. In this section, we always assume the numerical condition $dm = p^e - 1$ for some positive integer $e$.

**Proposition 7.6** ([CdB21, Lemma 4.2])**.** *The Chern numbers of* $\Omega^1_{R_m}\left(\log \widetilde{C_d}\right)$ *are given by*

$$\begin{aligned} c_1^2\left(\Omega^1_{R_m}\left(\log \widetilde{C_d}\right)\right) &= d(m-3) - m^2 + 6 \\ c_2\left(\Omega^1_{R_m}\left(\log \widetilde{C_d}\right)\right) &= m^2 + 1. \end{aligned}$$

*In particular,*

$$\lim_{d \to \infty} \frac{c_1^2(\Omega^1_{R_m}(\log \widetilde{C_d}))}{c_2(\Omega^1_{R_m}(\log \widetilde{C_d}))} = \infty.$$

*Proof.* We note that $R_m$ is the blow up of $\mathbb{P}^2$ at $m^2$ points, hence $K^2_{R_m} = 9 - m^2$

and $c_2(R_m) = m^2 + 3$. It follows that

$$\begin{aligned} c_1^2\left(\Omega^1_{R_m}\left(\log \widetilde{C_d}\right)\right) &= \left(K_{R_m} + \widetilde{C_d}\right)^2 \\ &= K_{R_m}^2 + 2K_{R_m}\cdot \widetilde{C_d} + \widetilde{C_d}^2 \\ &= 9 - m^2 + 2d(m-3) - 2 + d(3-m) - 1 \\ &= d(m-3) - m^2 + 6. \\ c_2\left(\Omega^1_{R_m}\left(\log \widetilde{C_d}\right)\right) &= c_2(R_m) + \widetilde{C_d}\cdot\left(K_R + \widetilde{C_d}\right) \\ &= m^2 + 3 - \deg K_{\widetilde{C_d}} \\ &= m^2 + 1. \end{aligned}$$

□

**Proposition 7.7** ([CdB21, Lemma 4.3])**.** *We have*

$$\chi\left(R_m, 2K_{R_m} + 2\widetilde{C_d}\right) = d(m-3) - m^2 + 5.$$

*Moreover, if $m > 3$ and $\chi\left(R_m, 2K_{R_m} + 2\widetilde{C_d}\right) > 0$, then we have $h^0\left(R_m, 2K_{R_m} + 2\widetilde{C_d}\right) > 0$.*

*Proof.* By the Riemann-Roch theorem,

$$\begin{aligned} &\chi\left(R_m, 2K_{R_m} + 2\widetilde{C_d}\right) \\ &= \frac{1}{2}\cdot 2\left(K_{R_m} + \widetilde{C_d}\right)\cdot\left(K_{R_m} + 2\widetilde{C_d}\right) + \chi\left(R_m, \mathcal{O}_{R_m}\right) \\ &= \left(K_{R_m} + \widetilde{C_d}\right)^2 + \deg K_{\widetilde{C_d}} + 1 \\ &= d(m-3) - m^2 + 5, \end{aligned}$$

which proves the first claim.
Next, we assume $m > 3$ and $\chi\left(R_m, 2K_{R_m} + 2\widetilde{C_d}\right) > 0$. By the Serre duality,

$$h^2\left(R_m, 2K_{R_m} + 2\widetilde{C_d}\right) = h^0\left(R_m, -K_{R_m} - 2\widetilde{C_d}\right).$$

The latter has to be zero, otherwise it will follow that $h^0\left(R_m, -K_{R_m}\right) > 0$ since $\widetilde{C_d}$ is effective, contradicting $K_{R_m} = \mathcal{O}_{R_m}(3-m, 1)$ by Lemma 5.2. Hence $h^2\left(R_m, 2K_{R_m} + 2\widetilde{C_d}\right) = 0$, and $\chi\left(R_m, 2K_{R_m} + 2\widetilde{C_d}\right) > 0$ implies $h^0\left(R_m, 2K_{R_m} + 2\widetilde{C_d}\right) > 0$. □

Hence Theorem 2.4 fails on the surface $R_m$ if $m > 3$, and there is no bound for $c_1^2\left(\Omega^1_{R_m}\left(\log \widetilde{C_d}\right)\right)/c_2\left(\Omega^1_{R_m}\left(\log \widetilde{C_d}\right)\right)$ if we let $d$ vary.

### 7.2 Harbourne constants

The study of Harbourne constants gives a possible approach to one direction of Question 5 in the Introduction: *Let $\widetilde{X} \to X$ be the blow up of $X$ at $n$ disjoint points, can we derive bounded negativity of $\widetilde{X}$ from the bounded negativity of $X$?* First note that the converse direction is true by Proposition 3.4.
To define the Harbourne constant of a surface, we first show that the irreducibility hypothesis in the Bounded Negativity Conjecture can be removed.

**Proposition 7.8** ([BHK+13, Porposition 5.1])**.** *For a surface $X$, there exists a constant $b(X)$ such that $C^2 \geq -b(X)$ for all negative curves if and only if there exists a constant $b'(X)$ such that $C^2 \geq -b'(X)$ for all reduced curves.*

*Proof.* Let $C$ be a reduced curve. We take the Zariski decomposition $C \sim_{\mathbb{Q}} P+N = \sum_i a_i P_i + \sum_j b_j N_j$, where $P_i, N_j$ are irreducible components of $P, N$. Since the intersection matrix $(N_i \cdot N_j)_{i,j}$ is negative definite, the $N_j$'s are linearly independent in $N^1(X)$. Moreover, $b_j \leq 1$ since $C$ is reduced. By the Hodge index theorem, the number of $N_j$ is at most $\rho(X)-1$. We then have

$$C^2 = \left(\sum_i a_i P_i + \sum_j b_j N_j\right)^2 \geq \sum_j N_j^2 \geq (\rho(X)-1)\cdot(-b(X)).$$

□

**Definition 7.9.** Let $\mathcal{P} = \{p_1, \ldots, p_n\}$ be a set of distinct points on a surface $X$. Let $\widetilde{X}$ be the blow up of $X$ at $\mathcal{P}$. For a reduced curve $C$ on $X$ we write $\widetilde{C}$ for the strict transform of $C$ on $\widetilde{X}$. We define the Harbourne constant of $X$ at $\mathcal{P}$ as

$$H(X;\mathcal{P}) := \inf_C \frac{\widetilde{C}^2}{n},$$

where the infimum is taken over all the reduced curves on $X$.

Equivalently, the Harbourne constant of $X$ at $\mathcal{P}$ can be defined as $H(X;\mathcal{P}) = \inf_C \frac{C^2}{n} - \frac{1}{n}\sum_{i=1}^n m_i^2$, where $m_i$ is the multiplicity of $C$ at $m_i$.

**Definition 7.10.** Let $X$ be a surface. We define the global Harbourne constant of $X$ as

$$H(X) := \inf_{\mathcal{P}} H(X;\mathcal{P}),$$

where the infimum is taken over all finite sets of distinct points on $X$.

It follows then directly from definition that if $H(X) > -\infty$, then $X$ has bounded negativity. Moreover, if $X$ satisfies $H(X) > -\infty$, we can deduce that any blow up of $X$ at distinct points also has bounded negativity.

**Proposition 7.11** ([BDRH$^+$15, Remark 2.3])**.** *Let $X$ be a surface having bounded negativity such that $H(X) > -\infty$. Then, for any blow up $\widetilde{X}$ of $X$ at finitely many distinct points, $\widetilde{X}$ also has bounded negativity.*

*Proof.* If $\widetilde{X}$ is the blow up of $X$ at $\mathcal{P} = \{p_1, \ldots, p_n\}$, there are two types of integral curves on $\widetilde{X}$: the exceptional divisors $E_1, \ldots, E_n$, and the strict transform $\widetilde{C}$ of an integral curve $C$ on $X$. We have the following inequalites for $\widetilde{C}^2$:

$$\widetilde{C}^2 \geq nH(X; \mathcal{P}) \geq nH(X).$$

For the exceptional divisors $E_i$ we have $E_i^2 = -1$, hence the result follows. □

We do not use any reducible curves here, but generalizing the definition to all reduced curves relates the bounded negativity to configuration of curves on surfaces. We take here the projective plane as an example. It is an old question in curve configurations on $\mathbb{P}^2$ asking: *How good are the multiplicities of a curve at some points bounded, if the degree of the curve is bounded?* If the global Harbourne constant of $\mathbb{P}^2$ is not $-\infty$, we can bound the average multiplicity at $n$ points of a curve of degree $d$ as follows:

$$\frac{1}{n}\sum_{i=1}^{n} m_i \leq \sqrt{\frac{1}{n}\sum_i m_i^2} \leq \sqrt{\frac{d^2}{n} - H(\mathbb{P}^2)}.$$

In the case of positive characteristic, we know that $H(\mathbb{P}^2) = -\infty$, since the surface $R_m$ in Theorem 5.1 is a blow up of $\mathbb{P}^2$ and does not have bounded negativity. Now by Proposition 7.11, we obtain $H(\mathbb{P}^2) = -\infty$. In the case of characteristic 0, we still do not know whether $H(\mathbb{P}^2)$ is bounded. An upper bound of $H(\mathbb{P}^2)$ known now is the Wiman configuration (See [Wim96]) which contains 45 lines and gives $\frac{1}{n}\left(C^2 - \sum_i m_i^2\right) = -225/67$.
For more detailed discussions about the Harbourne constants we refer to [DHS21] and [BDRH$^+$15].

## 7.3 Weighted Bounded Negativity Conjecture

In the Weighted Bounded Negativity Conjecture we hope to bound the "weighted self intersection" instead of the self intersection, which is weaker than the original conjecture. The precise formulation is as follows.

**Conjecture 7.12** (Weighted Bounded Negativity Conjecture)**.** *For each smooth, projective surface $X$, there exists a number $b(X)$ such that $\frac{C^2}{(C \cdot L)^2} \geq -b(X)$ for all irreducible and reduced curves $C$ on $X$, and all big and nef line bundles $L$ with $C \cdot L > 0$.*

The validity of the Weighted Bounded Negativity Conjecture implies the strict positivity of local Seshadri constant, which is related to the study of the nef cone of blow ups.

**Definition 7.13.** Let $X$ be a surface and $x \in X$ be a point. Let $f : \widetilde{X} \to X$ be the blow up of $X$ at $x$. Denote the exceptional divisor with $E$. Let $H$ be an ample line bundle on $X$. We define the local Seshadri constant of $H$ at $x$ to be

$$\varepsilon(X, H, x) := \sup_{\varepsilon \in \mathbb{R}} \{\varepsilon > 0 \mid f^*H - \varepsilon E \text{ is nef}\}.$$

Or equivalently,

$$\varepsilon(X, H, x) := \inf \left\{ \frac{H \cdot C}{m_C} \;\middle|\; \begin{array}{l} C \text{ irreducible reduced curve passing through } x, \\ m_c \text{ the multiplicity of } C \text{ at } x \end{array} \right\}.$$

We define the local Seshadri constant at $x$ as

$$\varepsilon(X, x) := \inf_{H \text{ ample}} \varepsilon(X, H, x).$$

It is still unknown whether the local Seshadri constant is strictly positive in general. But we can show that the Weighted Bounded Negativity Conjecture implies the strict positivity of the local Seshadri constant.

**Proposition 7.14** ([BBC+12, Proposition 3.7.2])**.** *If the Weighted Bounded Negativity Conjecture is true, then $\varepsilon(X, x) > 0$ for all surfaces $X$ and points $x \in X$.*

*Proof.* Let $f : \widetilde{X} \to X$ be the blow up of $X$ at $x$. Let $b(\widetilde{X})$ be the bound given by the Weighted Bounded Negativity Conjecture. Let $C$ be an irreducible reduced curve on $X$ with multiplicity $m$ at $x$. Let $H$ be an ample line bundle on $X$. Then $f^*H$ is big and nef, and $f^*H \cdot \widetilde{C} = H \cdot C > 0$, where $\widetilde{C}$ is the strict transform of $C$ under $f$. We deduce that

$$C^2 = \widetilde{C}^2 + m^2 \geq -b(\widetilde{X}) \cdot (H \cdot C)^2 + m^2.$$

Let $H, L_1, \ldots, L_n$ be an orthogonal basis of $N^1(X)$. By the Hodge Index Theorem, we obtain $L_i^2 < 0$. Let $C = a_0 H + \sum_i a_i L_i$ be the decomposition of $C$ in $N^1(X)$. We obtain

$$C^2 \cdot H^2 = \left(a_0^2 H^2 + \sum a_i^2 L_i^2\right) \cdot H^2 \leq a_0^2 \left(H^2\right)^2 = (C \cdot H)^2.$$

This implies

$$\frac{(C \cdot H)^2}{m^2} \geq \frac{C^2 \cdot H^2}{m^2} \geq (1 - \frac{b(\widetilde{X})(C \cdot H)^2}{m^2}) \cdot H^2,$$

which is equivalent to

$$\frac{1 + b(\widetilde{X}) \cdot H^2}{m^2} \cdot (C \cdot H)^2 \geq H^2.$$

Hence we deduce that

$$\frac{C \cdot H}{m} \geq \sqrt{\frac{H^2}{1 + b(\widetilde{X}) \cdot H^2}} = \sqrt{\frac{1}{\frac{1}{H^2} + b(\widetilde{X})}} \geq \frac{1}{\sqrt{1 + b(\widetilde{X})}},$$

which finishes the proof. □

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